COUNTING UNROOTED MAPS ON THE PLANE

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Abstract. A planar map is a 2-cell imbedding of an undirected connected graph, loops and parallel edges allowed, on the sphere. A plane map is a planar map with a distinguished outside face. A rooted map is a map with a distinguished oriented edge and an unrooted map is an equivalence class of maps under orientation-preserving homeomorphism which, in the case of a plane map, fixes the distinguished face. Previously we obtained formulae for the number unrooted planar maps of various classes, including all maps, non-separable maps, eulerian maps, loopless maps and unicursal maps with n edges; P. Leroux, G. Labelle and M. Bousquet did so for unrooted planar and plane maps with two faces. In this article we obtain formulae for counting rooted and unrooted plane maps of all these classes and their duals except unicursal maps. The formulae for rooted maps are all sum-free; the formulae for unrooted maps contain at most a sum over the divisors of n. For rooted unicursal maps and their duals we find a counting formula containing a sum over the integers from 0 to n-2. Numerical tables for all these plane maps, rooted and unrooted, with up to 20 edges are provided in an appendix.

0. INTRODUCTION

A *map* is a 2-cell imbedding of an undirected connected graph, loops and parallel edges allowed, on an unbounded surface. If the surface is a sphere, then the map is a *planar map*; if the surface is an infinite plane, then the map is a *plane map* and one of its faces is distinguished as the *outside face*. Given any planar map or plane map, the number v of its vertices, the number n of its edges and the number f of its faces satisfies Euler's formula:

$$v + f = n + 2. \qquad (0.1)$$

A *dart* of a map on an orientable surface is a half-edge or edge-end. A *homeomorphism* between two maps on orientable surfaces is bicontinuous bijection between their imbedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other; in the case of plane maps, it also takes the outside face of one map into the outside face of the other. An *isomorphism* between two maps on orientable surfaces is an equivalence class of orientation-preserving homeomorphisms, where two homeomorphisms are considered equivalent if they induce the same bijection between the darts of one map and the darts of the other, and an *automorphism* of a map is an isomorphism between the map and itself. A map is *rooted* by distinguishing a dart, called the *root*. The only automorphism of a map that fixes the root is the trivial automorphism that fixes all its darts [Tu63]. The following proposition is easily proved from the above definitions:

Proposition 0.1: Distinguishing each of the f faces of a rooted planar map gives rise to f distinct rooted plane maps.□

An *isthmus* is an edge that is incident on both sides to the same face; thus, an isthmus is the face-vertex dual of a loop. The *degree* of a vertex is the number of edges incident to it, loops counting twice. The *degree* of a face is the number of edges on its boundary, isthmuses counting twice.

An *unrooted* map is an isomorphism class of maps. The number of unrooted planar maps with n edges was found by V.A. Liskovets in 1978 and published in [Li81]. Other classes of unrooted planar maps that have been counted by number of edges are non-separable [LiWa83], eulerian (all vertices of even degree) and unicursal (exactly two vertices of odd degree) [LiWa04a], loopless [LiWa04b], one-face (trees) [Wk72] and two-face (plane as well as planar) [BoLaLr00]. The class of all planar maps and of non-separable planar maps are *self-dual*: for every map in the class, its face-vertex dual is also in the class. The remaining classes listed above are not self-dual; the corresponding dual classes are bipartite (all faces of even degree), dual-unicursal (exactly two faces of odd degree), isthmusless and two-vertex maps. By face-vertex duality there are exactly as many rooted or unrooted n-edge maps in a non-self-dual class as there are in its dual class.

In this article we find a relationship between the formulae for $M^+(n)$, $M^+_f(n)$ and $M^+_v(n)$, the number of unrooted n-edge planar and plane maps of a given class and plane maps of the dual class, respectively (distinguishing a face in the dual class is equivalent to distinguishing a vertex in the primal class, whence the notation $M^+_v(n)$). The formula for $(M^+_f(n)+M^+_v(n))/2 - M^+(n)$, or for $M^+_f(n) - M^+(n)$ if the class is self-dual, turns out to be considerably simpler than any of the three original formulae, so that if any two of them are known (either one of them in the case of a self-dual class), then the remaining one is easy to calculate. The general formulae are derived in Section 1 and then applied to obtain formulae for the number of all plane maps and non-separable plane maps (Section 2), eulerian and bipartite plane maps (Section 3), loopless and isthmusless plane maps (Section 4) and two-vertex plane maps (Section 5). Numerical tables are given in an appendix.

1. PLANE VS. PLANAR

The following general scheme for counting unrooted planar n-edge maps comes from [Li81], to which the reader is referred for details.

Let $M'(n)$ and $M^+(n)$ be the number of rooted and unrooted n-edge maps, respectively, of a given class. By Burnside's lemma,

$$2nM^+(n) = \sum_{\rho} \text{fix}(n,\rho), \tag{1.1}$$

where fix(n,ρ) is the number of rooted maps *fixed by* ρ - that is, for which ρ is an automorphism of the unrooted version of the map - and ρ runs over all the permutations of the $2n$ darts which can be an automorphism of an n-edge map.

The identity permutation, which fixes all the darts, is an automorphism of every n-edge map; so the contribution of this permutation to (1.1) is

$$M'(n). \tag{1.2}$$

Any non-trivial automorphism ρ of a planar map can be represented as a rotation of the sphere about an axis that intersects the map in two *elements* (vertices, edges or faces) called *axial elements*. A rooted map Γ fixed by an automorphism of period p can be represented as p isomorphic copies of a rooted map Δ, called the *quotient map* of Γ with respect to ρ and denoted by Γ/ρ. To each non-axial element of Δ there correspond p non-axial elements of Γ and to each of the two axial elements of Δ there corresponds a single axial element of Γ whose degree is p times the number of darts in the corresponding axial element of Δ. An axial edge of Δ has only one dart; so to make Δ a map we complete this half-edge with a vertex of degree 1 called a *singular vertex*; the single dart contained by the singular vertex was added along with the vertex and is therefore not the root. Given a rooted map Δ with at most two singular vertices, two elements chosen to be axial (which are either vertices or faces and must include all the singular vertices) and a non-trivial automorphism ρ (which must be of period 2 if Δ contains at least one singular vertex), there is a unique rooted map Γ such that $\Delta=\Gamma/\rho$; so fix(n,ρ) is equal to the number of rooted maps that are the quotient maps of some rooted n-edge map with respect to ρ.

For i=0, 1 and 2, let $Q'_i(n)$ be the number of rooted quotient maps with i singular vertices of all the rooted n-edge maps of a given class. If i=1, then the quotient map has $(n+1)/2$ edges, so that n must be odd. If i=2, then the quotient map has $(n+2)/2$ edges, so that n must be even. Substituting these values and (1.2) into (1.1) we obtain the following general formula:

$$2nM^+(n) = M'(n) + Q'_0(n) + \begin{cases} Q'_1(n) \text{ if } n \text{ is odd,} \\ Q'_2(n) \text{ if } n \text{ is even.} \end{cases} \qquad (1.3)$$

Now let $M^+_f(n)$ be the number of unrooted n-edge maps of a given class with a distinguished face. The analogue of (1.1) is

$$2nM^+{}_f(n) = \sum_\rho \text{fix}(n,\rho), \qquad (1.4)$$

where ρ must also fix the distinguished face.

Suppose that ρ is the trivial automorphism. The analogue of (1.2) is the number of rooted n-edge maps of the same class with a distinguished face. By Proposition 0.1 and formula (0.1), this is

$$\sum_{v+f=n+2} fM'(v,f), \qquad (1.5)$$

where $M'(v,f)$ is the number of rooted maps of that class with v vertices and f faces. This number is equal to the total number of faces in all the n-edge rooted maps of that class.

For i=0, 1 and 2, let $Q'_{f,i}(n)$ be the number of rooted quotient maps with i singular vertices of all the rooted n-edge maps of the given class under a non-trivial automorphism one of whose axial elements is the distinguished face. If i=1, then as in the case of planar maps n must be odd. If i=2, then as before n must be even; also, since both of the axial elements are the singular vertices, neither of them can be the distinguished face, so that $Q'_{f,2}(n)=0$. The analogue of (1.3) is thus

$$2nM^+{}_f(n) = \sum_{v+f=n+2} fM'(v,f) + Q'_{f,0}(n) + \begin{cases} Q'_{f,1}(n) \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even.} \end{cases} \qquad (1.6)$$

The number of unrooted n-edge maps in the dual class with a distinguished face is equal to the number $M^+_v(n)$ of unrooted n-edge maps in the primal class with a distinguished vertex. For i=0, 1 and 2, let $Q'_{v,i}(n)$ be the number of rooted quotient maps with i singular vertices of all the rooted n-edge maps of the given class under a non-trivial automorphism one of whose axial elements is the distinguished vertex. Using the face-vertex dual of Proposition 0.1 we obtain the following analogue of (1.6):

$$2nM^+{}_v(n) = \sum_{v+f=n+2} vM'(v,f) + Q'_{v,0}(n) + \begin{cases} Q'_{v,1}(n) \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even.} \end{cases} \qquad (1.7)$$

Proposition 1.1 For any class of maps,

$$Q'_{f,0}(n)+Q'_{v,0}(n)=2Q'_0(n). \tag{1.8}$$

Proof. The number $Q'_0(n)$ is the total number of ways of choosing an unordered pair of axial elements, each of which may be either a face or a vertex, in all the rooted planar maps that are the quotient maps with no singular vertices of some rooted planar n-edge map of the given class. Since the maps are rooted, all unordered pairs of elements are distinct. If the axial elements are now labeled north and south, the pairs of axial elements are now ordered, increasing the number of pairs to $2Q'_0(n)$. If the north axial element is a vertex, then it can be called the distinguished vertex, and the total number ways of distinguishing a vertex, making it the north axial element and then choosing the south axial element is $Q'_{v,0}(n)$. If the north axial element is a face, then it can be called the distinguished face, and the total number of ways of distinguishing a face, making it the north axial element and then choosing the south axial element is $Q'_{f,0}(n)$. Then (1.8) follows from the fact that the north axial element must either be a vertex or a face.□

Proposition 1.2. For any class of maps,

$$Q'_{f,1}(n)+Q'_{v,1}(n)=Q'_1(n). \tag{1.9}$$

Proof. The number $Q'_1(n)$ is the total number of ways of choosing the axial element that isn't the singular vertex in all the rooted planar maps that are the quotient maps with one singular vertex of some rooted planar n-edge map of the given class. If the non-singular axial element is a vertex, then it can be called the distinguished vertex, and the total number of ways of distinguishing a vertex is $Q'_{v,1}(n)$. If the non-singular element is a face, then it can be called the distinguished face, and the total number of ways of distinguishing a face is $Q'_{f,1}(n)$. Then (1.9) follows from the same fact as (1.8).□

Proposition 1.3. For any class of maps,

$$2n\left(M_f^{\,+}(n)+M_v^{\,+}(n)\right) = (n+2)M'(n)+2Q'_0(n)+\begin{cases} Q'_1(n) \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even.} \end{cases} \tag{1.10}$$

Proof. This formula follows trivially from formulae (1.6), (1.7), (1.8) and (1.9).□

Proposition 1.4. For any self-dual class of maps,

$$4nM_f^{\,+}(n) = (n+2)M'(n)+2Q'_0(n)+\begin{cases} Q'_1(n) \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even.} \end{cases} \tag{1.11}$$

Proof. This formula follows trivially from (1.10) and the fact that for a self-dual class of maps, $M^+_v(n)=M^+_f(n)$.□

Comparing (1.10) and (1.11) with (1.3) we see that we can eliminate $Q'_0(n)$ by subtracting twice formula (1.3) from either (1.10) or (1.11). The respective formulae, divided by the coefficient of $M^+_f(n)$, are

$$M_f^{\,+}(n)+M_v^{\,+}(n) = 2M^+(n)+\frac{1}{2}M'(n)-\begin{cases} Q'_1(n)/2n \text{ if } n \text{ is odd,} \\ Q'_2(n)/n \text{ if } n \text{ is even} \end{cases} \tag{1.12}$$

and

$$M_f^{+}(n) = M^{+}(n) + \frac{1}{4}M'(n) - \begin{cases} Q'_1(n)/4n \text{ if } n \text{ is odd,} \\ Q'_2(n)/2n \text{ if } n \text{ is even.} \end{cases} \tag{1.13}$$

Since $Q'_0(n)$ generally contains a sum over divisors of n whereas $Q'_1(n)$ is generally a single term, eliminating $Q'_0(n)$ rather than $Q'_1(n)$ leads to a formula that is at once more elegant and more computationally more efficient if we have a table of values of $M^+(n)$.

2. ALL MAPS AND NON-SEPARABLE MAPS

The number of unrooted planar n-edge maps is given by the following formula [Li81] :

$$2nA^{+}(n) = A'(n) + \sum_{t|n, t<n} \phi\left(\frac{n}{t}\right)\binom{t+2}{2}A'(t) + \begin{cases} \frac{n(n+3)}{2}A'\left(\frac{n-1}{2}\right) \text{ if } n \text{ is odd,} \\ \frac{n(n-1)}{2}A'\left(\frac{n-2}{2}\right) \text{ if } n \text{ is even,} \end{cases} \tag{2.1}$$

where $A'(n)$, the number of rooted planar n-edge maps, is given by the following formula [Tu63] :

$$A'(n) = \frac{2 \times 3^n \times (2n)!}{n!(n+2)!}. \tag{2.2}$$

Since the class of all planar n-edge maps is self-dual, we obtain from (1.11) and (2.1) the following formula for the number $A^+_f(n)$ of unrooted plane n-edge maps :

$$4nA_f^{+}(n) = (n+2)A'(n) + 2 \times \sum_{t|n, t<n} \phi\left(\frac{n}{t}\right)\binom{t+2}{2}A'(t) + \begin{cases} \frac{n(n+3)}{2}A'\left(\frac{n-1}{2}\right) \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even,} \end{cases} \tag{2.3}$$

From (1.13) we obtain the following expression for $A^+_f(n)$ in terms of $A^+(n)$:

$$A_f^{+}(n) = A^{+}(n) + \frac{1}{4}A'(n) - \begin{cases} \frac{n+3}{8}A'\left(\frac{n-1}{2}\right) \text{ if } n \text{ is odd,} \\ \frac{n-1}{4}A'\left(\frac{n-2}{2}\right) \text{ if } n \text{ is even.} \end{cases} \tag{2.4}$$

The number of unrooted non-separable planar n-edge maps is given by the following formula [LiWa83] :

$$2nB^{+}(n) = B'(n) + \sum_{t|n, t<n} \phi\left(\frac{n}{t}\right)\left(1 + 9\binom{t}{2}\right)B'(t) + \begin{cases} \frac{n(n+1)}{2}B'\left(\frac{n+1}{2}\right) \text{ if } n \text{ is odd,} \\ \frac{n(3n-4)}{8}B'\left(\frac{n}{2}\right) \text{ if } n \text{ is even,} \end{cases} \tag{2.5}$$

where $B'(n)$, the number of rooted non-separable planar n-edge maps, is given by the following formula [Tu63] :

$$B'(n) = \frac{2\times(3n-3)!}{n!(2n-1)!}. \tag{2.6}$$

Since this class too is self-dual, we obtain from (1.11) and (2.5) the following formula for the number $B^{+}{}_{f}(n)$ of unrooted non-separable plane n-edge maps:

$$4nB_f{}^{+}(n) = (n+2)B'(n) + 2\times\sum_{t|n,t<n}\phi\left(\frac{n}{t}\right)\left(1+9\binom{t}{2}\right)B'(t) + \begin{cases}\frac{n(n+1)}{2}B'\left(\frac{n+1}{2}\right) \text{ if } n \text{ is odd,}\\ 0 \text{ if } n \text{ is even.}\end{cases} \tag{2.7}$$

From (1.13) we obtain the following expression for $B^{+}{}_{f}(n)$ in terms of $B^{+}(n)$:

$$B_f{}^{+}(n) = B^{+}(n) + \frac{1}{4}B'(n) - \begin{cases}\frac{n+1}{8}B'\left(\frac{n+1}{2}\right) \text{ if } n \text{ is odd,}\\ \frac{3n-4}{16}B'\left(\frac{n}{2}\right) \text{ if } n \text{ is even.}\end{cases} \tag{2.8}$$

3. EULERIAN AND BIPARTITE MAPS

The number of rooted eulerian planar maps with n edges [Wa75] is

$$E(n) = \frac{3\times 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}. \tag{3.1}$$

From this formula and a result of [Li85] it was shown in [LiWa04a] that the number of unrooted eulerian plamar maps with n edges is given by

$$2nE^{+}(n) = \frac{3\times 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n} + 3\sum_{k<n,k|n}\phi\left(\frac{n}{k}\right)2^{k-2}\binom{2k}{k} + \begin{cases}\frac{n\times 2^{(n+1)/2}}{n+1}\binom{n-1}{(n-1)/2} \text{ if } n \text{ is odd,}\\ \sum_{k|n/2}\phi\left(\frac{n}{k}\right)2^{k-2}\binom{2k}{k} + \frac{n\times 2^{(n-2)/2}}{n+2}\binom{n}{n/2} \text{ if } n \text{ is even.}\end{cases} \tag{3.2}$$

The first term on the right side of (3.2) is contributed by the trivial automorphism, the second term by the non-trivial automorphisms under which the quotient maps are eulerian, the term for odd n by quotient maps with one singular vertex, the first term for even n by unicursal quotient maps with no singular vertices and the remaining term by quotient maps with two singular vertices.

Suppose now that a face is distinguished and called the north axial element. The following proposition was proved in [KoLiWa00].

Proposition 3.1. The number of rooted eulerian planar maps with n edges and a distinguished vertex is

$$(n+2)E'(n)/3. \square \tag{3.3}$$

By (1.5) and Euler's formula (0.1), number of rooted eulerian planar maps with n edges and a distinguished face is equal to $(n+2)E'(n)-(n+2)E'(n)/3$. By (3.1) we have

$$\sum_{v+f=n+2} fM'(v,f) = \frac{2^n}{n+1}\binom{2n}{n}. \tag{3.4}$$

For each value of k in the second term of the right side of (3.2), the coefficient of $\phi(n/k)$ is equal to $E'(k)$ multiplied by the number of choices of axial pairs, which is $(k+2)(k+1)/2$ because any one of the $k+2$ vertices and faces can be chosen as an axial element and the axial elements are not distinguished. But now we are distinguishing a face and calling it the north axial element. If a given quotient map has v vertices, then it has $f=k+2-v$ faces. Since the north axial element must be the distinguished face, there are $k+2-v$ ways of distinguishing the face and calling it the north axial element. The south axial element can then be chosen from any of the $k+1$ vertices and other faces, so that the number of choices of ordered axial pairs is $(k+1)(k+2-v)$. By an argument similar to the derivation of (3.4), we replace the factor $((k+1)(k+2)/2)E'(k)$ in the second term on the right side of (3.2) by $(k+1)(k+2)E'(k)-(k+1)(k+2)E'(k)/3$ to obtain

$$\sum_{k<n,k|n} \phi\left(\frac{n}{k}\right)2^k\binom{2k}{k}. \tag{3.5}$$

The remaining terms on the right side of (3.2) are all contributed by quotient maps whose axial elements are vertices, either singular or non-singular; so none of these terms contribute to the number of eulerian maps with a distinguished (axial) face; so $Q'_{f,0}(n)$ is given by formula (3.5) and $Q'_{f,1}(n)=0$. Substituting these values into (1.6) we obtain

Proposition 3.2. The number of unrooted eulerian plane maps with n edges is given by

$$2nE_f{}^+(n) = \frac{2^n}{n+1}\binom{2n}{n} + \sum_{k<n,k|n} \phi\left(\frac{n}{k}\right)2^k\binom{2k}{k}. \tag{3.6}$$

From (1.10), (3.1), (3.2) and (3.6) we obtain

Corollary 3.3. The number of unrooted bipartite plane maps with n edges is given by

$$2nE_v{}^+(n) = \frac{2^{n-1}}{n+1}\binom{2n}{n} + \sum_{k<n,k|n} \phi\left(\frac{n}{k}\right)2^{k-1}\binom{2k}{k} + \begin{cases} \dfrac{n\times 2^{(n+1)/2}}{n+1}\dbinom{n-1}{(n-1)/2} & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{k|n/2} \phi\left(\frac{n}{k}\right)2^{k-1}\binom{2k}{k} & \text{if } n \text{ is even.} \end{cases} \tag{3.7}$$

From (1.12), (3.1) and (3.2) we obtain the following formula:

$$E_f^{+}(n) + E_v^{+}(n) = 2E^{+}(n) + \frac{1}{2}E(n) - \begin{cases} \dfrac{2^{(n-1)/2}}{n+1}\dbinom{n-1}{(n-1)/2} & \text{if } n \text{ is odd}, \\ \dfrac{2^{(n-2)/2}}{n+2}\dbinom{n}{n/2} & \text{if } n \text{ is even}. \end{cases} \tag{3.8}$$

4. LOOPLESS AND ISTHMUSLESS MAPS

The number of rooted loopless planar maps with n edges [WaLe75] is

$$L'(n) = \frac{2 \times (4n+1)!}{(n+1)!(3n+2)!}. \tag{4.1}$$

From this formula it was shown in [LiWa04b] that the number of unrooted loopless planar maps with n edges is given by

$$2nL^{+}(n) = \frac{2 \times (4n+1)!}{(n+1)!(3n+2)!} + \sum_{k<n, k|n} \phi\left(\frac{n}{k}\right)\binom{4k}{k} + \begin{cases} \dfrac{2n}{n+1}\dbinom{2n}{(n-1)/2} & \text{if } n \text{ is odd}, \\ \dbinom{n}{(n-2)/2} & \text{if } n \text{ is even}. \end{cases} \tag{4.2}$$

The first term on the right side of (4.2) is contributed by the trivial automorphism, the second term by the non-trivial automorphisms under which the quotient maps are loopless, the term for odd n by quotient maps with one singular vertex and the term for even n by quotient maps with two singular vertices.

Suppose now that a face is distinguished and called the north axial element. The following proposition was proved in [KoLiWa00].

Proposition 4.1. The number of rooted loopless planar maps with n edges and a distinguished vertex is

$$(n+2)L'(n) - \frac{(4n)!}{(3n+1)!\,n!}. \tag{4.3}$$

The following corollary follows trivially from Proposition 4.1 and Euler's formula.

Corollary 4.2. The number of rooted loopless planar maps with n edges and a distinguished face is

$$\sum_{v+f=n+2} fM'(v,f) = \frac{(4n)!}{(3n+1)!\,n!}. \tag{4.4}$$

We now evaluate the remaining terms in (1.6) specialized to loopless maps, the distinguished-face analogue of (4.2), by following the argument used in [LiWa04b], modifying it wherever necessary to account for the distinguished face. The quotient map of a loopless map under a non-trivial automorphism is either a loopless map or a nested sequence of loopless maps $M_1,\ldots,M_k$ with each pair of adjacent components of the sequence separated by a loop. In the latter case, one axial element is in the extremal components M_1 with a edges and the other axial element is in the other extremal component M_k with b edges; also, an axial element is not allowed to be the vertex in its component incident to the loop separating that component from the adjacent one in the sequence. We suppose for the moment that the quotient map has n edges; later we will substitute the appropriate number of edges into the enumeration formula for rooted quotient maps we obtain below.

Suppose that the quotient map has no singular vertices.

If the quotient map is not loopless, then the number of such maps *without a distiguished face and with the axial elements not distinguished from each other* is given by formula (4.5), which is a corrected version of formula (16) of [LiWa04b] in which it is not assumed that $a \geq b$ (it is (4.5) which leads to the enumeration formula obtained in [LiWa04b]).

$$n \sum_{a+b \leq n} (a+1)L'(a)(b+1)L'(b) \times [x^{n-(a+b)-1}](1+z), \tag{4.5}$$

where

$$z = x(1+z)^4 \tag{4.6}$$

and $[x^i]$(<expression>) means the coefficient of x^i in that expression.

We modify (4.5) to account for the fact that the axial elements are now distinguished from one another and that one of them is the distinguished face. Distinguishing the axial elements multiplies (4.5) by 2. Without loss of generality we call the extremal component containing the distinguished face M_k, which has b edges. In (4.5), the factor $(b+1)L'(b)$ was obtained by taking all the b-edge rooted loopless maps and choosing any of the $b+1$ faces or vertices except the forbidden one (incident to the loop) to be the axial element. Instead, we choose any face of M_k to be the axial element; so that $(b+1)L'(b)$ must be replaced by the total number of faces in all the rooted loopless maps with b edges, which is given by (4.4) with n replaced by b.

We reproduce formula (17) of [LiWa04b] (with a replaced by b) as (4.7).

$$\sum_{b=0}^{\infty} (b+1)L'(b)x^b = (1+z)^2. \tag{4.7}$$

The analgous formula that must replace (4.7) is given in the following proposition.

Proposition 4.3.

$$\sum_{b=0}^{\infty} \frac{(4b)!}{(3b+1)!\,b!} x^b = (1+z). \tag{4.8}$$

Proof. The formula for Lagrange inversion (see [La81] for a combinatorial proof) in the special case when $z=xg(z)$ (instead of the general case $z=a+xg(z)$) can be simplified to

$$[x^0]f(z)=f(0);\ [x^n]f(z)=(1/n)[z^{n-1}](f'(z)(g(z))^n) \text{ for all } n\geq 1. \qquad (4.9)$$

Here $g(z)$ is given by (4.6) as $(1+z)^4$. Equating the coefficient of x^n in the left side of (4.8) with (4.9) we find that $f'(z)=1$, so that $f(z)=z+C$, where C is some constant. Since the coefficient of x^0 in the left side of (4,8) is 1, by the first equation of (4.9) we have $f(0)=1$, whence we obtain (4.8).□

Comparing (4.7) with (4.8) and recalling that we must multiply by 2, we see that instead of formula (18) of [LiWa04b], which is evaluated from (4.5) and is equal to $n[x^{n-1}](1+z)^5$, we must instead use

$$2n[x^{n-1}](1+z)^4. \qquad (4.10)$$

Applying Lagrange inversion to (4.10) we obtain the formula

$$8\frac{n}{n-1}\times[z^{n-2}](1+z)^{4n-1}=2n\frac{(4n)!}{(3n+1)!\,n!}. \qquad (4.11)$$

If the quotient map is loopless, then the north axial element is the distinguished face and the south axial element can be any of the other $n+1$ vertices and faces; so the number of these quotient maps is given by

$$(n+1)\frac{(4n)!}{(3n+1)!\,n!}. \qquad (4.12)$$

Adding (4.11) to (4.12) we obtain the total number of n-edge quotient maps with no singular vertices of loopless maps, given by formula (4.13):

$$\binom{4n}{n}. \qquad (4.13)$$

Now if the automorphism is of order n/k, and there are $\phi(n/k)$ such automorphisms, then the quotient map will have k edges, so that $Q'_{f,0}(n)$ (see formula (1.6)) is equal to

$$Q'_{f,0}(n)=\sum_{k<n,k|n}\phi\left(\frac{n}{k}\right)\binom{4k}{k}, \qquad (4.14)$$

which is also equal to $Q'_0(n)$, the second term on the right side of (4.2). A bijective proof of this equality would be interesting.

Suppose now that the quotient map has one singular vertex. Again for the moment we suppose the quotient map to have n edges.

Suppose the quotient map is not loopless. Then the number of such maps *without a distinguished face but with the south axial element being the distinguished face* is given by (4.15) which is formula (23) of [LiWa04b]:

$$\frac{2n-1}{2}\sum_{a+b\le n-1}(2a+1)L'(a)(b+1)L'(b)\times[x^{n-(a+b)-2}](1+z). \tag{4.15}$$

Since the axial elements are already distinguished, to account for the distinguished face we need not multiply by 2; we just replace $(b+1)L'(b)$ by (4.4) with n replaced by b, which means that instead of simplifying (4.15) to

$$\frac{2n-1}{2}\times[x^{n-2}](1+z)^6, \tag{4.16}$$

which is formula (25) of [LiWa04b], we instead obtain

$$\frac{2n-1}{2}\times[x^{n-2}](1+z)^5. \tag{4.17}$$

Applying Lagrange inversion to (4.17) and simplifying, we obtain the formula

$$\frac{5(2n-1)(4n-4)!}{(n-2)!(3n-1)!}. \tag{4.18}$$

Suppose the quotient map is loopless. Without the distinguished face, the number of such maps is given by formula (4.19), which is formula (28) of [LiWa04b]:

$$n(2n\text{-}1)L'(n\text{-}1). \tag{4.19}$$

The factor n represents the number of choices of the north axial element. To account for the distinguished face, we replace $nL'(n\text{-}1)$ by (4.4) with n replaced by n-1 and obtain

$$\frac{(2n-1)(4n-4)!}{(n-1)!(3n-2)!}. \tag{4.20}$$

Adding (4.20) to (4.18) we obtain the total number of n-edge quotient maps with one singular vertex of loopless maps, given by formula (4.21):

$$\binom{4n-2}{n-1}. \tag{4.21}$$

Now the quotient map of an n-edge map will have not n edges but $(n+1)/2$. By replacing n by $(n+1)/2$ in (4.21), we find that

$$Q'_{f,1}(n)=\binom{2n}{(n-1)/2}. \tag{4.22}$$

Substituting from (4.4), (4.14) and (4.22) into (1.6), we obtain

Proposition 4.4. The number $L_f^+(n)$ of unrooted loopless plane maps with n edges is

$$2nL_f^{+}(n) = \frac{(4n)!}{(3n+1)!\,n!} + \sum_{k<n, k|n} \phi\left(\frac{n}{k}\right)\binom{4k}{k} + \begin{cases} \binom{2n}{(n-1)/2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4.23}$$

To obtain the number $L_v^{+}(n)$ of unrooted isthmusless plane maps with n edges, we use formula (1.10) with the letter M replaced by L everywhere. Now $2nL_f^{+}(n)$ is given by (4.21), $L'(n)$ by (4.1), and the remaining terms of (1.10) are the corresponding terms of (4.2). Making these substitutions we obtain

Corollary 4.5. The number of unrooted isthmusless plane maps with n edges is

$$2nL_v^{+}(n) = \frac{(5n^2+13n+2)\times(4n)!}{(3n+2)!(n+1)!} + \sum_{k<n, k|n} \phi\left(\frac{n}{k}\right)\binom{4k}{k} + \begin{cases} \left(\frac{n-1}{n+1}\right)\binom{2n}{(n-1)/2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4.22}$$

From (1.12) and (4.2) we obtain

$$L_f^{+}(n) + L_v^{+}(n) = 2L^{+}(n) + \frac{1}{2}L'(n) - \begin{cases} \binom{2n}{(n-1)/2}/(n+1) & \text{if } n \text{ is odd,} \\ \binom{n}{(n-2)/2}/n & \text{if } n \text{ is even.} \end{cases} \tag{4.23}$$

5. TWO-VERTEX AND OTHER MAPS

There are other classes of maps, aside from the ones treated above, for which unrooted enumeration in the plane can be easily obtained by a slight modification of the methods we designed for the sphere. These include triangular (or, dually, trivalent) maps, which we leave as an open problem, two-face maps, which were treated in [BoLaLr00], and two-vertex maps, which we treat below.

From [BoLaLr00] we have the following formulae (taken, in order, from formulae (16), (78) and (15)).

The number of rooted 2-face planar maps is

$$T'(n) = 2^{2n-1} - \binom{2n-1}{n-1}. \tag{5.1}$$

The number of unrooted 2-face planar maps is given by

$$2nT^{+}(n) = T'(n) + \sum_{k<n, k|n} \phi\left(\frac{n}{k}\right)T'(k) + \begin{cases} n\binom{n-1}{(n-1)/2} & \text{if } n \text{ is odd,} \\ n\binom{n-1}{n/2} & \text{if } n \text{ is even.} \end{cases} \tag{5.2}$$

The number of unrooted 2-face plane maps is given by

$$2nT_f^{+}(n) = 2T'(n) + 2\sum_{k<n,k|n} \phi\left(\frac{n}{k}\right)T'(k). \tag{5.3}$$

To calculate $T_v^+(n)$ from these formulae and (1.10) we have to know $Q'_0(n)$ and $Q'_1(n)$. These numbers are contained in [BoLaLr00]; we recalculate them independently. The second term on the right side of (5.2) is the total contribution made by the non-trivial automorphisms that fix both faces. The axial elements are the two faces and cannot be singular vertices, so that $Q'_1(n)$ is equal to the last term in (5.2) when n is odd. But $Q'_0(n)$ is greater than the second term on the right side of (5.2) because some of the quotient maps with no singular vertices are contributed by automorphisms that switch the two faces. In this case the quotient map is a rooted plane tree with $n/2$ edges and, therefore, $n/2 + 1$ vertices. Both the axial elements are vertices; so the contribution of the face-switching automorphisms to $Q'_0(n)$ is given by the number of rooted plane trees with $n/2$ edges (the Catalan number with index $n/2$ [HaPrTu64]) multiplied by the number of unordered pairs of vertices chosen from among $n/2 + 1$. It follows that

$$Q'_0(n) = \sum_{k<n,k|n} \phi\left(\frac{n}{k}\right)T'(k) + \begin{cases} 0 \text{ if } n \text{ is odd,} \\ \dfrac{n}{2}\dbinom{n-1}{n/2} \text{ if } n \text{ is even.} \end{cases} \tag{5.4}$$

Substituting from (5.1), (5.3), (5.4) and the term of (5.2) that is equal to $Q'_1(n)$ into (1.10), we obtain

Proposition 5.1. The number $M_v^+(n)$ of unrooted 2-vertex plane maps is given by

$$2M_v^{+}(n) = 2^{2n-1} - \binom{2n-1}{n-1} + \begin{cases} \dbinom{n-1}{(n-1)/2} \text{ if } n \text{ is odd,} \\ \dbinom{n-1}{n/2} \text{ if } n \text{ is even.} \end{cases} \tag{5.5}$$

Now $Q'_2(n)$ is equal to the last term on the right side of (5.2) for even n minus the corresponding term in (5.4). Substituting for $Q'_1(n)$ and $Q'_2(n)$ into (1.12) we obtain

$$M_f^{+}(n) + M_v^{+}(n) = 2M^{+}(n) + \frac{1}{2}M'(n) - \begin{cases} \dfrac{1}{2}\dbinom{n-1}{(n-1)/2} \text{ if } n \text{ is odd,} \\ \dfrac{1}{2}\dbinom{n-1}{n/2} \text{ if } n \text{ is even} \end{cases} \tag{5.6}$$

It is no coincidence that (5.5) contains no sum over the divisors of n; a non-trivial automorphism that fixes a vertex must exchange the two faces and thus be of order 2. If (5.3) had not been available, it would have been easier to derive (5.5) directly and then obtain (5.3) from (5.5) using (1.12) instead of the other way around.

Not all classes of maps yield closed form-formula for rooted enumeration in the plane even if they do so on the sphere. To illustrate this point, we compare the enumeration of rooted unicursal planar maps done in [LiWa04a] with the enumeration of rooted unicursal plane maps which we do below.

A map is called *unicursal* if exactly two of its vertices are of odd degree. The number $U'(n)$ of rooted unicursal planar maps with n edges was shown in [LiWa04a] to be equal to

$$U'(n) = [x^{n-1}]\sum_{v=2}^{n+1}\frac{n!}{(v-2)!(n-v+2)!}(1-4x)^{-1}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-2}. \tag{5.7}$$

Setting $z=x(z+1)^2$ and using Lagrange inversion, we evaluated (5.7) as

$$U'(n) = 2\frac{(2n-1)!}{(n+1)!(n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i} + \frac{(2n)!}{(n+1)!(n-1)!}\sum_{i=0}^{n-2}\binom{n-2}{i}, \tag{5.8}$$

which we simplified to

$$U'(n) = 2^{n-2}\binom{2n}{n},\ n \geq 1. \tag{5.9}$$

The number $U'_f(n)$ of rooted unicursal plane maps with n edges is found by multiplying each term in the sum of (5.7) by the number n-v+2 of faces:

$$U'_f(n) = [x^{n-1}]\sum_{v=2}^{n+1}\frac{n!}{(v-2)!(n-v+1)!}(1-4x)^{-1}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-2}. \tag{5.10}$$

Setting $z=x(z+1)^2$ and using Lagrange inversion, we evaluate (5.10) as

$$U'_f(n) = 2\frac{(2n-1)!}{(n-1)!(n-1)!}\sum_{i=0}^{n-2}\frac{\binom{n-2}{i}}{n+1+i} + \frac{(2n)!}{(n-1)!(n-1)!}\sum_{i=0}^{n-2}\frac{\binom{n-2}{i}}{n+2+i}. \tag{5.11}$$

This formula is valid only for $n\geq 2$; $U'_f(1)=1$ because there is only one rooted unicursal map with one edge and it has one face.

A map is called *dual-unicursal* if exactly two of its faces are of odd degree. The number $U'_v(n)$ of rooted dual-unicursal plane maps is given by the following formula:

$$U'_v(n) + U'_f(n) = (n+2)U'(n). \tag{5.12}$$

Unlike the sums in (5.8), the sums in (5.11) do not seem to simplify; Maple evaluated them in terms of hypergeometric functions. An interesting open problem would be to find a closed-form formula for $U'_f(n)$ or to prove that none exists. A more interesting open problem would be to find a systematic method for deciding whether a closed-form formula exists for the number of rooted planar or plane n-edge maps of a given class by examining the maps themselves instead of the result of a Lagrange inversion.

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APPENDIX: NUMERICAL TABLES

Table 1a: The number of rooted plane maps and non-separable maps

edges	all maps	non-separable maps
1	3	3
2	18	2
3	135	5
4	1134	18
5	10206	77
6	96228	364
7	938223	1836
8	9382230	9690
9	95698746	52877
10	991787004	296010
11	10413763542	1690845
12	110546105292	9817080
13	1184422556700	57769740
14	12791763612360	343806368
15	139110429284415	2065802056
16	1522031755700070	12515350122
17	16742349312700770	76367432013
18	185047018719324300	468922828150
19	2054021907784499730	2895381678735
20	22887672686741568420	17966214519330

Table 1b: The number of unrooted plane maps and non-separable maps

edges	all maps	non-separable maps
1	2	2
2	6	1
3	26	2
4	150	4
5	1032	10
6	8074	37
7	67086	138
8	586752	628
9	5317226	2972
10	49592424	14903
11	473357994	76994
12	4606116310	409594
13	45554761836	2222628
14	456848968518	12281570
15	4637014782748	68864086
16	47563495004742	391120036
17	492422043299964	2246122574
18	5140194991046122	13025721601
19	54053208147441474	76194378042
20	572191817441284272	449155863868

Table 2a: The number of rooted plane eulerian maps and bipartite maps

edges	eulerian maps	bipartite maps
1	2	1
2	8	4
3	40	20
4	224	112
5	1344	672
6	8448	4224
7	54912	27456
8	366080	183040
9	2489344	1244672
10	17199104	8599552
11	120393728	60196864
12	852017152	426008576
13	6085836800	3042918400
14	43818024960	21909012480
15	317680680960	158840340480
16	2317200261120	1158600130560
17	16992801914880	8496400957440
18	125210119372800	62605059686400
19	926554883358720	463277441679360
20	6882979133521920	3441489566760960

Table 2b: The number of unrooted plane eulerian maps and bipartite maps

edges	eulerian maps	bipartite maps
1	1	1
2	3	2
3	8	5
4	32	18
5	136	72
6	722	368
7	3924	1982
8	22954	11514
9	138316	69270
10	860364	430384
11	5472444	2736894
12	35503288	17752884
13	234070648	117039548
14	1564945158	782480424
15	10589356592	5294705752
16	72412611194	36206357114
17	499788291616	249894328848
18	3478059566250	1739030128872
19	24383023246284	12191512867814
20	172074483068320	86037243899240

Table 3a: The number of rooted plane loopless maps and isthmusless maps

edges	loopless maps	isthmusless maps
1	1	2
2	4	8
3	22	43
4	140	268
5	969	1824
6	7084	13156
7	53820	98865
8	420732	765948
9	3362260	6075256
10	27343888	49094708
11	225568798	402801425
12	1882933364	3346590068
13	15875338990	28099903160
14	134993766600	238079915640
15	1156393243320	2032914717645
16	9969937491420	17476713955548
17	86445222719724	151143219598008
18	753310723010608	1314045772469632
19	6594154339031800	11478299163026540
20	57956002331347120	100688538612524720

Table 3b: The number of unrooted plane loopless maps and isthmusless maps

edges	loopless maps	isthmusless maps
1	1	1
2	2	3
3	6	9
4	22	38
5	103	187
6	614	1120
7	3872	7083
8	26414	47990
9	186988	337676
10	1367976	2455517
11	10254326	18310155
12	78461338	139447034
13	610598818	1080773098
14	4821248244	8502896424
15	38546510368	67763884363
16	311560875422	546147639926
17	2542507084588	4445389286380
18	20925300483992	36501274080076
19	173530381632724	302060508150976
20	1448900079476152	2517213486505592

Table 4a: The number of rooted plane two-face and two-vertex maps

edges	two-face maps rooted	2-vertex maps
1	2	1
2	10	10
3	44	66
4	186	372
5	772	1930
6	3172	9516
7	12952	45332
8	52666	210664
9	213524	960858
10	863820	4319100
11	3488872	19188796
12	14073060	84438360
13	56708264	368603716
14	228318856	1598231992
15	918624304	6889682280
16	3693886906	29551095248
17	14846262964	126193235194
18	59644341436	536799072924
19	239532643144	2275560109868
20	961665098956	9616650989560

Table 4b: The number of unrooted plane two-face and two-vertex maps

edges	two-face maps	2-vertex maps
1	1	1
2	3	3
3	8	12
4	25	48
5	78	196
6	270	798
7	926	3248
8	3305	13184
9	11868	53416
10	43232	216018
11	158586	872344
12	586530	3518496
13	2181088	14177528
14	8154710	57080572
15	30620868	229657792
16	115435625	923474944
17	436654794	3711572176
18	1656793374	14911097514
19	6303490610	59883185096
20	24041649128	240416320928

Table 5: The number of rooted plane unicursal and dual-unicursal maps

edges	unicursal maps	dual-unicursal maps
1	1	2
2	10	14
3	93	107
4	836	844
5	7355	6757
6	63750	54522
7	546553	441863
8	4646920	3589880
9	39250935	29206025
10	329789450	237780982
11	2758868981	1936486411
12	22995369996	15771410420
13	191074697203	128431734797
14	1583463268366	1045618229234
15	13092015636465	8510270668815
16	108024564809744	69241255165936
17	889730213085167	563154350637073
18	7316434446188562	4578526894227438
19	60078376613838829	37209886138826771
20	492692533579612180	302291556342169580