COUNTING MAPS ON DOUGHNUTS

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Abstract. How many maps with V vertices and E edges can be drawn on a doughnut with G holes? I solved this problem for doughnuts with up to 10 holes, and my colleagues Alain Giorgetti and Alexander Mednykh counted maps by number of edges alone on doughnuts with up to 11 holes. This expository paper outlines, in terms meant to be understandable by a non-specialist, the methods we used and those used by other researchers to obtain the results upon which our own research depends.

Keywords: rooted maps; unrooted maps; orientable surfaces; exact enumeration; generating functions; orbifolds

Section 1. Definitions

This article is neither a classical research paper nor a classical survey paper. There are many research and survey books and articles on map enumeration; two books that contain results and references to research papers in this area are [1] and [2]. It is, rather, an exposition of the methods my colleagues Alain Giorgetti and Alexander Mednykh and I used to count rooted and unrooted maps by genus and those used by other researchers to obtain the results upon which our own research depends. Detailed proofs are omitted because they are contained in other sources. I presented my most recent results, as well as those of Giorgetti and Mednykh, at the 8th French Combinatorial Conference in Orsay, France, and the three of us submitted a joint paper about our results to the proceedings of that conference. The article you are now reading borrows some material from the joint paper, but Prof. Srecko Brlek, who invited me to give this talk, assured me that an invited speaker has the privilege of repeating material from published and/or submitted articles. I would like to thank him for the invitation by treating him to his favourite type of doughnut.

When I am asked what sort of research I do, I sometimes reply that I count maps on doughnuts. If I am then asked why I chose such a frivolous research topic, I reply, "So that I can eat the doughnuts, of course!" If the truth be told, most of my research has been on other topics – a necessity when one works in a Computer Science Department – but my Ph.D. thesis [3] was on that topic and now, nearly 40 years later, I have returned to it with renewed appetite.

The mathematical name for a doughnut with G holes is a *closed, orientable surface of genus G without boundary*, and the holes are called *handles* rather than holes. Take a hollow sphere, cut two round holes in it and paste each end of a rubber tube to one of the holes. The tube becomes a handle by which the sphere can be carried. If you squeeze the sphere until its diameter is as small as the thickness of the tube, the sphere together with its handle looks a bit like a doughnut. A sphere with one handle is called a *torus* (see Figure 1). I don't know any special name for a sphere with more than one handle.

Sphere with handle. **Torus (doughnut with chocolate icing).**
Figure 1

A *genus-G map* is a 2-cell imbedding of a graph, loops and multiple edges allowed, on a surface of genus G, which in this article is assumed to be unbounded and orientable, with an orientation already ascribed to it (counter-clockwise, say). The vertices of the graph are points in the surface, an edge incident to two distinct vertices in the graph (a *link*) is an open curve whose endpoints are the corresponding points and an edge incident twice to a single vertex (a *loop*) is a closed curve containing the corresponding point. This is the only way in which a vertex can intersect with an edge, and two edges can never intersect. There may be more than one way to imbed a graph in a surface (see Figure 2).

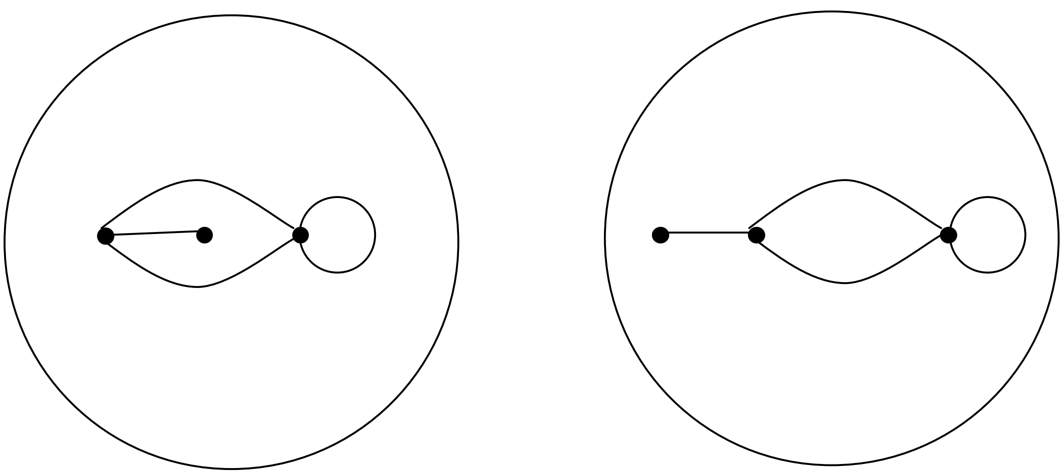
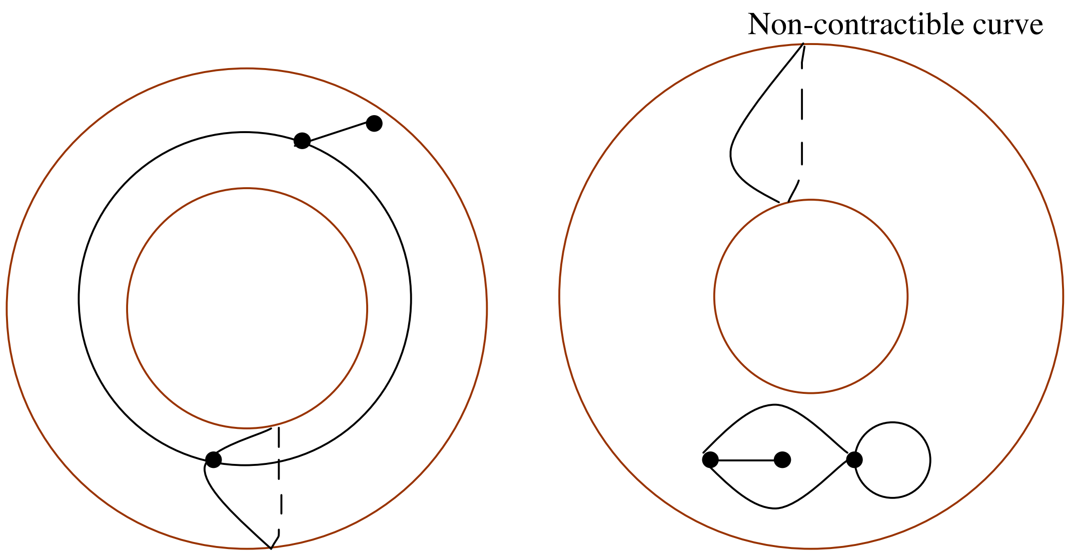


Figure 2: Same graph, two different maps on the sphere.

The connected components of the complement of the graph in the surface are called *faces* and they must be *simply connected*: that is, any closed curve contained in a face can be contracted to a point without touching the boundary of the face. This condition implies that the graph must be connected: otherwise you can draw a curve around one of the connected components and that curve can't be contracted to a point. The converse is true if the surface is a sphere, but not if it's any of the other surfaces (see the right-hand side of Figure 3).



Map on torus **Not a map (outside region not simply connected)**

Figure 3

For short, a map on a surface of genus G will be called a *genus-G map*. A *planar map* is a genus-0 map (a map on a sphere) and a *toroidal map* is a genus-1 map (a map on a torus or doughnut). If a map on a surface of genus G has V vertices, E edges and F faces, then by the Euler-Poincaré formula [4, chap. 9] (which you can verify for the maps drawn above)

$$F - E + V = 2(1\text{-}G). \tag{1}$$

A *homeomorphism* from one surface to another, or to itself, is a bijection that is bi-continuous; that is, it is continuous and so is its inverse. Note: every continuous deformation, like the one from which you can make a torus from a fat sphere with a skinny handle, is a homeomorphism, but not every homeomorphism is a continuous deformation. Two maps are *equivalent* if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other. A *dart* or *semi-edge* of a map or graph is half an edge. A loop is assumed to be incident twice to the same vertex, so that every edge, whether or not it is a loop, contains two darts. The *degree* of a vertex is the number of darts incident to it. The face incident to a dart d is the face incident to the edge containing d and on the right of an observer on d facing away from the vertex incident to d and the degree of a face is the number of darts incident to it. A *rooted map* is a map with a distinguished dart, its *root*. Two rooted maps are equivalent if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges, faces and the root of one map into the vertices, edges, faces and the root of the other.

A *combinatorial map* is a connected graph with a cyclic order imposed on the darts incident to each vertex, representing the order in which the darts of a (topological) map are encountered during a rotation around the vertex according to the orientation of the imbedding surface. The darts incident to a face are encountered by successive application of the following pair of actions: go from the current dart to the dart on the other end of the same edge and then to the next dart incident to the same vertex according to the cyclic order. In this way the faces of a combinatorial map can be counted, so that its genus can be calculated from (1). Two combinatorial maps are equivalent if they are related by a *map isomorphism* - a graph isomorphism that preserves this cyclic order - with an analogous definition for the equivalence of two rooted combinatorial maps. An *automorphism* of a combinatorial map is a map isomorphism from a map onto itself.

By *enumerating* maps with a given set of properties, whether rooted or not, we mean counting the number of equivalence classes of maps with these properties. It was shown in [5] that each equivalence class of topological maps is uniquely defined by an equivalence class of combinatorial maps; so for the purposes of enumeration, the term "map" can be taken to mean "combinatorial map". The number of rootings of a map is equal to the number of its darts (which is twice the number of its edges) divided by the number of automorphisms (see Figure 4).

This map has 2 automorphisms and thus 4 rootings. These 2 rootings are equivalent.
Figure 4

To count unrooted maps of genus G by any method that I know of (except generating them) you first have to be able to count rooted maps of every genus up to G. Section 2 contains a summary of the enumeration of rooted maps, both by number of edges and by number of faces and vertices, that had to be done so that unrooted maps could be counted. Section 3 contains a description of the enumeration of unrooted maps by number of edges. Section 4 contains a description of the enumeration of unrooted maps by number of edges and vertices. The Appendix contains tables of numbers of unrooted maps of genus 1, 2, 3, 4 and 5 with up to 11 edges, counted by number of edges and vertices.

2. Counting rooted maps

Rooted maps were introduced in [6] because they are easier to count than unrooted maps; this is because only the trivial map automorphism preserves the root [7], so that rooted maps can be counted without considering map automorphisms. Articles [6] and [7] were written by Prof. William Tutte, king of the graph theorists, whom I privately call King Tutte. He was trying to solve the four-colour problem, and he thought that if he could count planar maps and then count those planar maps whose vertices could be coloured in four colours so that no two adjacent vertices get the same colour, he could determine whether all planar maps could be coloured in this way. He never did solve that problem and it was eventually solved by other researchers using other methods. However, his work is of great value in its own right: among other contributions, he initiated map enumeration, which is now an active subject of research in many areas of mathematics, including classical and algebraic combinatorics [2], theoretical physics and integrable hierarchies [1].

In [6], Tutte found a closed-form formula for counting rooted planar maps with E edges:

$$N_0(E) = \frac{2 \times 3^E \times (2E)!}{E!(E+2)!}. \tag{2}$$

In [7], he counted rooted planar maps with V vertices and F faces: the number E of edges is obtained from (1). To do so, he took a rooted map with at least one edge and deleted the *root-edge*: the edge containing the root.

If the root-edge is incident to the same face on both sides, then the map will get disconnected and become an ordered pair of rooted planar maps with a total of E-1 edges, V vertices and F+1 faces (see Figure 5).

Figure 5: Deleting the root edge of a rooted planar map, disconnecting the map.

If the root-edge is incident to a different face on each side, the map becomes a rooted planar map with E-1 edges, V vertices and F-1 faces, but more than one rooted planar map can get transformed into the same rooted map (see Figure 6).

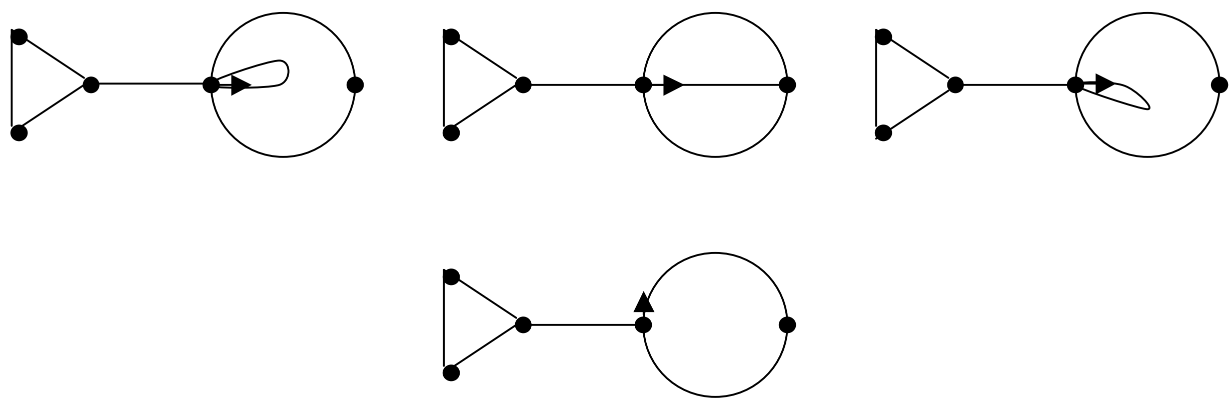


Figure 6: Deleting the root edge of a rooted planar map without disconnecting the map: this process is not uniquely reversible.

If the *root-face* (the face to the right of the root-edge to an observer on the root-edge facing away from the root) of the transformed map is of degree d, then the root-face of the original map can be anything from 1 up to d+1. In this way, he was able to express the number of rooted maps with V vertices and F faces, of which the root-face is of degree d, in terms of numbers of rooted maps with a given number of vertices, faces and degree of the root-face and with fewer edges than the original map. This sort of a formula is called a *recurrence*. From this recurrence he obtained a three-parameter generating function and then eliminated the parameter for the degree of the root-face to obtain a two-parameter generating function that counts rooted planar maps by number of vertices and faces. In [8] D. Arquès obtained a simpler two-parameter generating function counting the same set of objects.

For my Ph. D. thesis, written under the supervision of Prof. Alfred Lehman, I generalized Tutte's method to maps of higher genus. But first, since I was never much good with faces, I decided to use the face-vertex dual of deleting the root-edge. If the root-edge is a link (incident to a different **vertex** at each end), then the face-vertex dual of deleting it is contracting it to a point, merging its two incident vertices (see Figure 7). By analogy with deleting an edge incident to a different **face** on each side, you have to know the degree of the *root-vertex*: the vertex incident to the root.

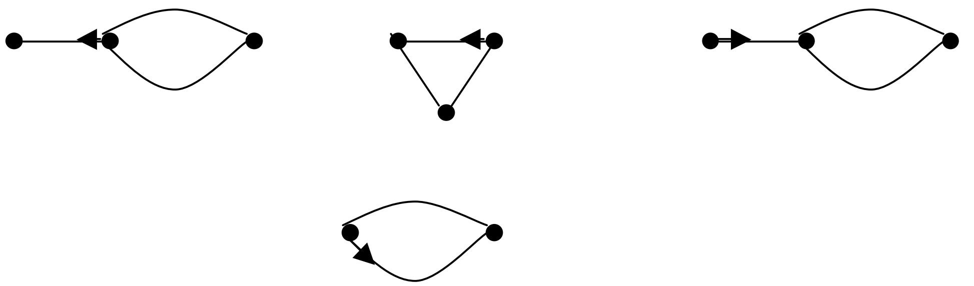


Figure 7: Contracting the root edge (a link) of a rooted map without disconnecting the map.

If the root-edge is a loop (incident to the same vertex at each end), then the face-vertex dual of deleting it is deleting it and splitting its incident vertex. If the map is planar, then this operation will disconnect it (see Figure 8).

Figure 8: Deleting the root edge (a loop) of a rooted map and splitting its incident vertex will disconnect the map if the map is planar.

If the map is not planar, then this operation may disconnect it into two components, the sum of whose genera is the genus of the original map (see Figure 9).

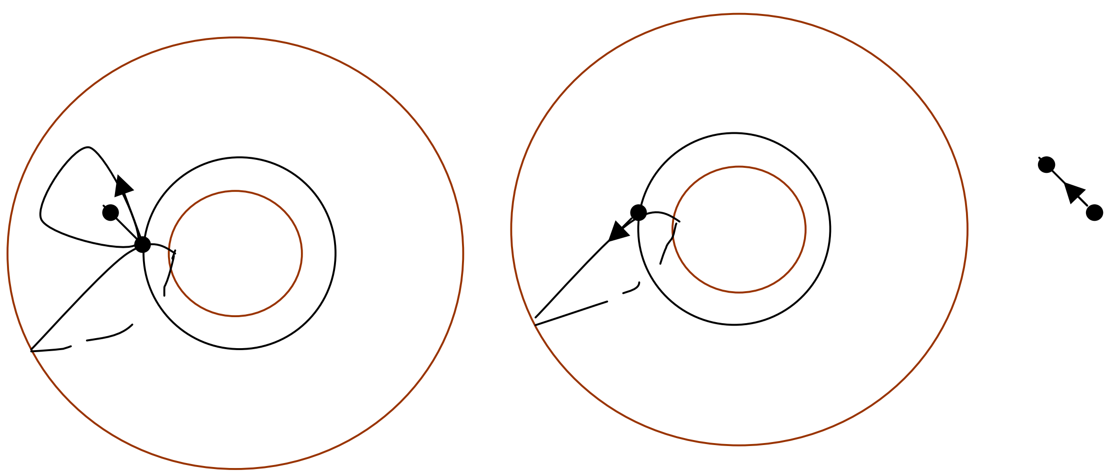


Figure 9: If the map is not planar, it could be split.

Or else it will reduce the genus of the map by 1 (see Figure 10), and now you have to know the degree of both of the new vertices to calculate the degree of the original one.

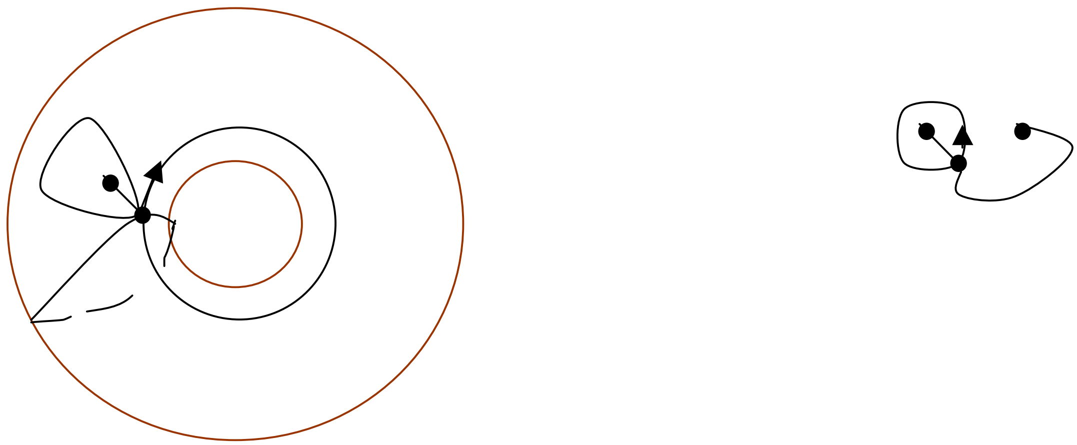


Figure 10: Or it could instead have its genus reduced by 1.

I chose a brute force solution to this problem: I kept track of the degrees of all the vertices, labelled them in decreasing order of degree and distinguished a dart incident to each vertex, treating the dart incident to the last vertex as the root. In this way I obtained a recurrence for the number of these *totally rooted maps*, which I unfortunately called dicings, as a function of the genus and the degree sequence. I made no attempt to convert my recurrence into a generating function. Instead, I solved it by computer up to 14 edges, which was as high as that 36-bit computer could go using the FORTRAN type INTEGER. For each genus G and degree sequence I multiplied the number of totally rooted maps by the appropriate factor to eliminate all but one distinguished dart and to account for the various permutations of the degree sequence, and then I summed over all the degree sequences with V terms that added up to $2E$ to get the number of rooted maps of genus G with E edges and V vertices. A table of the number of rooted genus-G maps with $E \leq 14$ edges and V vertices appears in my Ph. D. thesis [3]; a smaller table going up to 11 edges appears in the first of three papers, co-authored with Prof. Lehman, that contain the results of my thesis that are fit to read [9]. Of course, the algorithm I used to solve that recurrence is far from being polynomial-time. I later found a polynomial-time algorithm for counting rooted toroidal maps, both by number of edges alone and by number of vertices as well [10], but no explicit formula.

In [11], Edward Bender and E. Rodney Canfield introduced an improvement on the method of [3]. Since deleting a loop, decreasing the genus by 1, adds only one vertex whose degree you have to keep track of, to count rooted genus-G maps it is sufficient to know the degree of the first G+1 vertices and to distinguish a dart of only the first vertex as the root, thus reducing the number of maps that have to be considered. Now that case splits into two sub-cases, depending upon whether the piece of the split vertex that wasn't incident to the root is among those whose degree is known.

Using the method of [11], Didier Arquès [12] obtained a two-parameter generating function that counts rooted toroidal maps by number of vertices and faces. From this result, he obtained a closed-form formula for the number of rooted toroidal maps with E edges

$$N_1(E) = \sum_{k=0}^{E-2} 2^{E-3-k}(3^{E-1} - 3^k)\binom{E+k}{k} \tag{3}$$

and another one for the number of rooted toroidal maps with V vertices and F faces. In [13], Bender and Canfield obtained a generating function for the number of rooted maps of genus 2 and 3 with E edges.

In [14] Alain Giorgetti, a student of Arquès, generalized the results in [12] and [13] to obtain a general form for the generating function counting rooted maps of any genus $G > 0$ by number of vertices and faces. This generating function is the quotient of two functions; the numerator is a polynomial in two variables, representing the number of vertices and the number of faces. This polynomial is computed from a set of polynomials in several other variables, representing vertex degrees, that satisfy a very complicated recurrence he obtained. To make all these polynomials symmetric in all their variables, he distinguished a dart incident to each of the vertices whose degree is considered, which increases the size of the coefficients but does not increase the number of polynomials that have to be calculated. We note here that in the account of these results published in [15] the recurrence is missing a term; this omission was corrected in [16]. Programming in Maple 5, he solved his recurrence explicitly for $G = 2$ and $G = 3$ (these results are published in [15]) and he also computed the generating function that counts rooted maps of genus 4 by number of edges. The latter result was recently presented in [17], where it was used by Alexander Mednykh to count unrooted maps of genus 4 by number of edges (see Section 3). This was as far as the program written in Maple 5 was capable of carrying the calculations.

Recently, using Maple 13 and a more powerful computer, Giorgetti extended the solution of his recurrence up to genus 5. This is as far as the program written in Maple 13 was capable of carrying the calculations; it could have been optimized, but that was not its objective. The Maple code for counting rooted maps of a given genus is part of the MAP package [18].

I decided to try to extend the solution of Giorgetti's recurrence further. To improve the computational complexity of the calculations, I programmed mainly in C, using the library CLN (Class Library for Numbers) to handle big numbers - see the web site http://www.ginac.de/CLN/ - and the set of tools Xcode to run CLN - see the web site http://developer.apple.com/technologies/tools/xcode.html.
Jérôme Tremblay, a computer technician at my university, installed these two software packages on my computer and taught me how to write the C++ instructions that are needed to use them. And Giorgetti was of great help to me in debugging my program: he e-mailed me several intermediate polynomials satisfied by his recurrence. I optimized the solution to his recurrence and, once my program was finally debugged, I extended the enumeration by number of vertices and faces, as well as by number of edges, to genus 6. Although we each used a different algorithm and a different programming language, we both obtained the same answers, and the numbers of rooted maps we calculated agree with the tables in [3], providing evidence of the correctness of our results. These results are presented in [16].

More recently, Giorgetti solved the one-parameter version of his recurrence, which counts rooted maps of genus G by number of edges, up to genus 11, and I, using a more powerful computer, extended the enumeration of rooted maps by number of edges and vertices up to genus 10. Using these results, Medkykh, programming in Mathematica, counted unrooted maps of genus up to 11 by number of edges (see Section 3) and I, programming in C, counted unrooted maps of genus up to 10 by number of edges and vertices (see Section 4).

3. Counting unrooted maps by number of edges

An elegant formula for the number of unrooted planar maps with E edges was obtained by Valery Liskovets [19],[20]. He used the so-called Burnside's Lemma [21], which states that the number of orbits of a finite permutation group A acting on a set X is equal to the sum over all the elements a of A of the number of objects in X fixed by a divided by the cardinality of A. The set X is the set of darts of a map M with labelled darts. If M has E edges, it has $2E$ darts, so that $\#(A)=(2E)!$ A permutation a of the darts of M fixes M if a is an automorphism of the unlabelled version of M. However, the action of an automorphism of a map is completely determined by its action on a single dart [7]; so each rooted map can be dart-labelled in $(2E\text{-}1)!$ ways with the root getting the label 1. Applying this observation to Burnside's Lemma yields the following formula:

$$N^{+}(E) = \left(\sum_{a} \operatorname{fix}(a,E)\right)/(2E), \tag{4}$$

where $N^{+}(E)$ is the number of unrooted maps with E edges, a runs over all the permutations of the darts of a rooted map with E edges and $\operatorname{fix}(a,E)$ is the number of pairs (a,M), where M is a rooted map with E edges and a is an automorphism of the unrooted version of M.

A map automorphism is *regular* on its set of darts – that is, it is a permutation consisting of independent cycles of the same length [7]. One can thus consider only periodic orientation-preserving homeomorphisms of the embedding surface and call them automorphisms of the surface.

For the sphere, the non-trivial automorphisms are just rotations [22]. A rooted map M with an automorphism a of period L can thus be drawn on the sphere in such a way that the automorphism can be realized topologically by a rotation of period L. If the sphere is then sliced into L lunes and the lune containing the root inflated to a sphere, it will contain a rooted map with only $2E/L$ darts, the *quotient map* m of M under the automorphism a. The axis of rotation will pass through two distinct *cells* (vertices, edges or faces) of M, which we call its *poles*. If a pole is a *non-edge* (a face or a vertex), then it will be a non-edge of m (see Figure 11)

Figure 11: The quotient map of a planar map when neither pole is an edge.

If a pole is an edge, then it will be a *dangling* semi-edge of m – a semi-edge that is not part of a normal edge (see Figure 12).

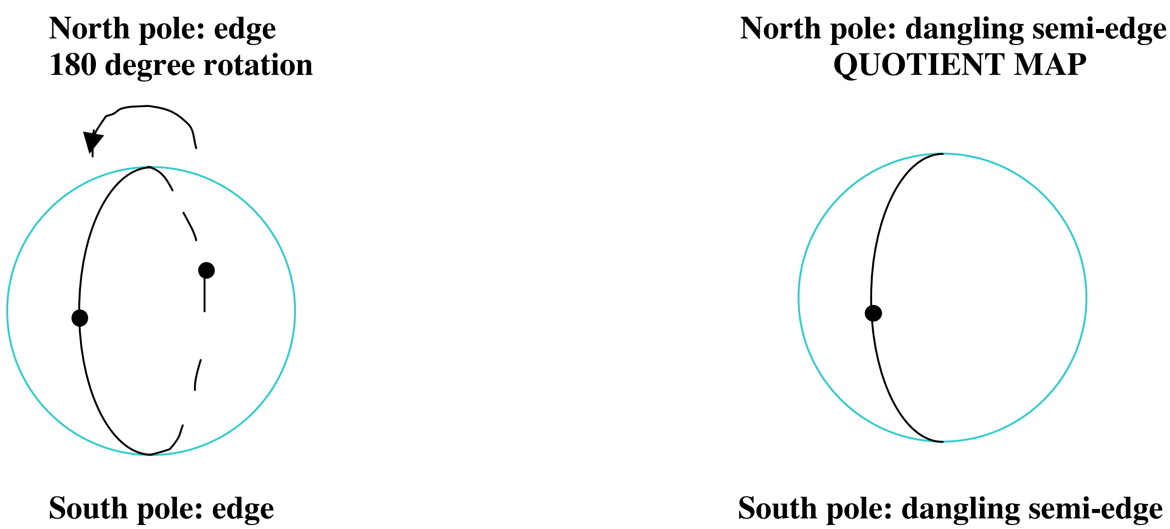


Figure 12: The quotient map of a planar map when at least one pole is an edge.

If at least one pole is an edge, then L must be 2, there is only one such automorphism, and m will have 1 or 2 dangling semi-edges. Once the axis of rotation has been chosen, the number of rotations of period L is $\phi(L)$, where ϕ is the Euler totient function, because a rotation of $(i/L)360$ degrees is of period L if and only if i and L are relatively prime.

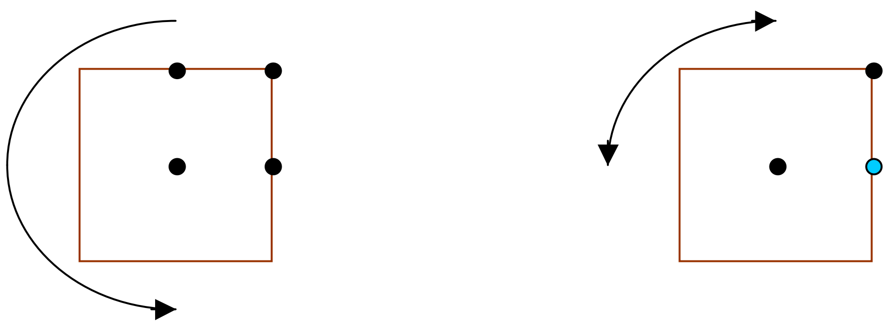
Using these ideas, Liskovets obtained the following formula for the number of unrooted planar maps with E edges:

$$N_0^+(E) = \frac{1}{2E}\left[N_0(E) + \sum_{t|E, t<E} \phi(E/t)\binom{t+2}{2} N_0(t)\right] + \begin{cases}((E+3)/4)\,N_0((E-1)/2), E \text{ odd}, \\ ((E-1)/4)\,N_0((E-2)/2), E \text{ even},\end{cases} \quad (5)$$

where $N_0(E)$ is the number of rooted planar maps with E edges and is given by (2).

Alexander Mednykh and Roman Nedela generalized Liskovets' result to surfaces of arbitrary orientable genus [23]. We illustrate their method on the torus.

We first represent a torus as a square with opposite sides identified in pairs, so that the four corners of the square represent a single point on the torus (see Figure 13). If the square is rotated by 180 degrees (L=2), there are four points on the torus that are fixed: the centre of the square, the point represented by the four corners and each of the two points represented by the middle of a pair of opposite sides. The automorphism thus has four orbits of length less than the period. Such an orbit is called a *branch point*, and the *branch index* of a branch point is L divided by the orbit length, in this case 2. There is one such automorphism. If instead the square is rotated by 90 degrees (L=4), the centre and the point represented by the four corners are fixed (two branch points of branch index 4) and the middle of both pairs of opposite sides are in a single orbit of length 2 (one branch point of branch index 2). There are two such automorphisms.

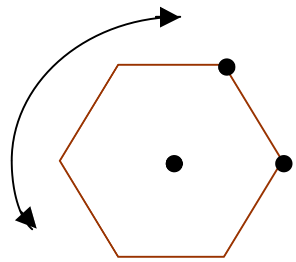


One 180-degree rotation,
4 branch points, each on a length-1 orbit.

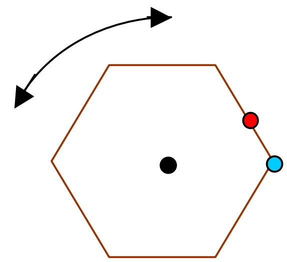
Two 90-degree rotations,
2 branch points, each on a length-1 orbit,
1 branch point on a length-2 orbit.

Figure 13: Automorphisms of period 2 and 4 of the torus, represented as a square.

Next we represent a torus as a hexagon with opposite sides identified in pairs, so that each triplet of non-adjacent corners represents a single point (see Figure 14). If the hexagon is rotated by 120 degrees (L=3), the centre is fixed and so is each triplet of non-adjacent corners (three branch points of branch index 3); there are two such automorphisms. If instead the hexagon is rotated by 60 degrees (L=6), the centre is fixed (a branch point of branch index 6), there is an orbit of length 2 (a branch point of branch index 3) containing the two triplets of non-adjacent corners and an orbit of length 3 (a branch point of branch index 2) containing the middle of all three pairs of opposite sides. There are two such automorphisms.



Two 120-degree rotations,
3 branch points, each on a length-1 orbit.



Two 60-degree rotations,
1 black branch point on a length-1 orbit,
1 blue branch point on a length-2 orbit,
1 red branch point on a length-3 orbit.

Figure 14: Automorphisms of period 3 and 6 of the torus, represented as a hexagon.

For all the above automorphisms, the quotient map is of genus 0.

There is also an infinite family of automorphisms with no branch points generated by two independent rotations: rotating the torus around the middle of the hole like the tube inside a tire on a spinning bicycle wheel and twisting it so that the valve no longer points to the centre of the wheel (see Figure 15). The number of such automorphisms of period L is $\phi_2(L)$, where $\phi_k(L)$ is the kth Jordan function of L, because if one rotation is through $(i/L)360$ degrees and the other is through $(j/L)360$ degrees, then for the combination to be of period L, GCD(i,j,L) must be 1. For all these automorphisms, the quotient map is of genus 1.

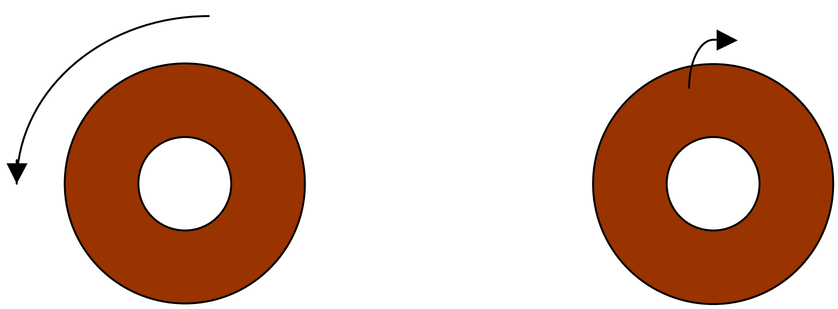


Figure 15: Automorphisms of the torus with no branch points.

In [23], the quotient space of a surface of genus G under an automorphism of some period L is called a *G-admissible orbifold*, and it is characterized by its *signature*: its genus g and its branch indices $m_1, \ldots, m_r$, where $1 < m_1 \le \cdots \le m_r$. Each such automorphism of period L corresponds to an (order-preserving) *epimorphism*: a surjection, with a torsion-free kernel, of the automorphism group of the surface of genus G onto the cyclic group Z_L of order L. In [23] the number of epimorphisms is given as a function of L and the signature by the formula

$$Epi_0\left(\pi_1(O), Z_L\right) = m^{2g}\phi_{2g}(L/m)E(m_1, \cdots, m_r), \qquad (6)$$

where $m = \mathrm{lcm}(m_1, \cdots, m_r)$, m divides L and

$$E(m_1, \cdots, m_r) = (1/m)\sum_{k=1}^{m} \prod_{i=1}^{r} \Phi(k, m_i), \tag{7}$$

where

$$\Phi(k,n) = \frac{\phi(n)}{\phi\big(n/\mathrm{gcd}(k,n)\big)} \mu\big(n/\mathrm{gcd}(k,n)\big)$$

is the von Sterneck function (see [24] for a discussion of the relation between this function and a certain trigonometric sum of Ramanujan). Here μ is the Möbius function, and in [23] the formula involving μ is used to express the Jordan function.

To avoid having to work with too many non-decreasing sequences of integers $(m_1, \ldots, m_r)$, the authors of [23] used Harvey's criterion [25] for the existence of a G-admissible orbifold with signature $[g;\ m_1, \ldots, m_r]$ under an automorphism of period L:

H1 (the Riemann-Hurwitz equation):

$$2 - 2G = L\left(2 - 2g - \sum_{i=1}^{r}\big(1 - (1/m_i)\big)\right), \tag{8}$$

H2: $m = \mathrm{lcm}(m_1, \cdots, m_r)$ divides L and $m=L$ if $g=0$,

H3: $\mathrm{lcm}(m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_r) = m$ for each $i=1,2,\ldots,r$,

H4: if m is even, the number of m_i divisible by the highest power of 2 that divides m is even,

H5: if $G\geq 2$, then $r\neq 1$ and $r\geq 3$ for $g=0$, if $G=1$ then $r\in\{0,3,4\}$, if $G=0$ then $r=2$.

They also bounded L using the Wiman theorem [26, page 131]: if $G>1$, then $1 \leq L \leq 4G+2$.

They then expressed the orbifold signature in terms of the number q_i of branch points with branch index i for every i from 2 to L and obtain the following formula for the number of rooted maps $\nu_O(d)$ with d darts on an orbifold $O = O[g; 2^{q_2}, \cdots, L^{q_L}]$:

$$\nu_O(d) = \sum_{s=0}^{q_2} \binom{d}{s} \binom{(d-s)/2 + 2 - 2g}{q_2 - s, q_3, \cdots, q_L} N_g\big((d-s)/2\big), \tag{9}$$

where $N_g(n)$ is the number of rooted maps of genus g with n edges (0 if n is not an integer). Here s is the number of dangling semi-edges in the quotient map m, all of which must be in branch points of branch index 2 so that they represent normal edges in the original map M. Given a rooted map with d-s darts, dangling semi-edges can be inserted into the slots between two adjacent darts according to the rotation about their common incident vertex, and since more than one dangling semi-edge can be inserted into the same slot, the number of ways to insert s dangling semi-edges is $\binom{d-1}{s}$. In addition, there are d ways to root the map once the dangling semi-edges have been inserted and only d-s ways to root it before the insertion. Multiplying the number of insertions by the ratio of the number of ways to root the two maps gives the binomial coefficient in the right side of (9). The multinomial coefficient in (9) is the number of ways in which the branch points with the various branch indices can be distributed among the non-edges of the quotient map; the "numerator" of the multinomial coefficient is the number of non-edges and is given by (1).

For each period L, if the original map has E edges, then the quotient map will have $2E/L$ darts. Substituting $2E/L$ for d in (9), multiplying by the number of epimorphisms corresponding to a given orbifold signature (6) and adding over all the orbifold signatures and then over all the periods L that divide E and finally dividing the sum by $2E$, as in (4), they obtained a formula for the number of unrooted maps of genus G with E edges:

$$\frac{1}{2E}\sum_{L|E}\sum_{O} Epi_0\big(\pi_1(O), Z_L\big)\nu_O(2E/L) \ , \tag{10}$$

where O runs over all the G-admissible orbifolds with period L and its signature is expressed as $[g;\ m_1,\ldots,m_r]$ when substituting into (6) and as $[g;2^{q_2},\cdots,L^{q_L}]$ when substituting into (9).

Of course, to get explicit numbers, they needed to know the number of rooted maps of genus up to G and the set of G-admissible orbifold signatures, along with the number of epimorphisms for each one. The former numbers were calculated from the formulas in [12] for G=1 and in [13] for G=2 and 3; the latter, for G≤4, were available from various sources. And so they were able to count unrooted maps of genus 1, 2 and 3 by number of edges and publish formulas and tables of numbers [23]. They also re-derived Liskovets' formula for the number of unrooted planar maps with E edges (5) as a special case of their own. This article contains enough information for you to do the same, and I recommend that you do so.

I met Mednykh and Nedela at a conference on Graph Embeddings and Maps on Surfaces (GEMS). Mednykh, in particular, was most anxious to extend the enumeration of unrooted maps by number of edges beyond genus 3, and he begged me to give him the generating function that counts rooted maps of genus 4 with E edges, flattering me by asserting that I was the only one who could calculate it. I didn't yet know how to do so; so I told him about the Arquès-Giorgetti paper [15], which, I said, contained everything he needed to calculate this generating function himself (I didn't yet know about the omission of a term from the recurrence). But he said he couldn't understand it: it is written in French and his knowledge of French is minimal. In the course of my frequent conversations by e-mail with Giorgetti, I learned that **he** did know how to calculate the required generating function; so I suggested to both Giorgetti and Mednykh that they collaborate to count both rooted and unrooted genus-4 maps by number of edges, which they did [17].

More recently, Giorgetti extended the enumeration of rooted maps by number of edges up to genus 11 and provided Mednykh with the appropriate generating functions. To count the corresponding unrooted maps, Mednykh also needed the set of G-admissible orbifolds and their number of epimorphisms up to G=11. These were provided by Ján Karabás, a student of Nedela, who computed them (programming in Magma) from Harvey's condition and formulas (6) and (7) for G up to 100 and made them available on his web site [27]. With these results and the generating functions provided by Giorgetti, Mednykh extended the enumeration of unrooted maps with E edges up to genus 11.

4. Counting unrooted maps by number of edges and vertices

At the GEMS conference, I expressed to Mednykh my intention of counting unrooted maps by number of edges and vertices. He asserted that it would be a lot of work, but it turned out to be easier than he thought, because most of the work had already been done. I had already applied Liskovets' method to obtain a formula for the number of unrooted planar maps with E edges and V vertices [28]. It is to be noted here that Nicholas Wormald counted planar maps by number of edges and vertices up to not only orientation-preserving isomorphism but also orientation-reversing isomorphism [29],[30]. The formula in [19] and [20] and the one in [28] have smaller asymptotic computational complexities than the respective algorithms in Wormald's papers restricted to counting unrooted planar maps up to orientation-preserving isomorphism. Liskovets suggested a way in which his method could be used to count unrooted planar maps up to both kinds of isomorphism [31]; finding an algorithm more efficient than Wormald's using the results in [31] is an interesting open problem.

The basic idea I used in [28] is to distribute the branch points (poles, in the case of planar maps) among the vertices, faces and dangling semi-edges of the quotient map instead of just among the non-edges and the dangling semi-edges. Suppose that the quotient map is of genus g and has v vertices, f faces and s dangling semi-edges. Then the number e of normal edges can be calculated from (1) and the number d of darts is $2e+s$. Suppose also that there are v_k branch points of orbit length k (orbit length = L divided by branch index) that contain a vertex and f_j branch points of orbit length j that contain a face. We denote by v_L and f_L the number of vertices and faces, respectively, that do not contain a branch point. The original map will have k vertices for every vertex in a branch point of orbit length k, L vertices for every vertex not in a branch point, j faces for every face in a branch point of orbit length j and L faces for every face not in a branch point. The total numbers V of vertices and F of faces in the original map are given by the formulas

$$V = \sum_{k=1}^{L} kv_k \tag{11}$$

and

$$F = \sum_{j=1}^{L} jf_j\,, \tag{12}$$

and the total number E of edges is equal to $L*d/2$.

The binomial coefficient in (9) doesn't change – it still represents the number of ways to insert s dangling semi-edges into a rooted map with d-s darts and the ratio of the number of rootings – but the multinomial coefficient in (9) must be modified. The number of ways to distribute the branch points among the vertices and faces is

$$\binom{v}{v_1,\cdots,v_L}\binom{f}{f_1,\cdots,f_L}. \tag{13}$$

For this number to be positive, the sum of all the numbers v_k cannot exceed v and the sum of all the f_j cannot exceed f; so v and f each start at their respective sums and increase by 1 until the number E of edges in the original map exceeds a user-defined maximum. With each increase of v or f, (13) gets updated using a single multiplication and division. This number, which replaces the multinomial coefficient in (9), must be computed for all sets of non-negative integers such that for each k, $v_k + f_k$ is equal to the total number of branch points of orbit length i that are not distributed to dangling semi-edges.

Once (9), modified as described above, is multiplied by the number of epimorphisms of the current period and orbifold signature, we get the contribution of that period, signature and the numbers v_k and f_j to $2E$ times the number of unrooted maps of genus G with E edges and V vertices. This contribution is added to the appropriate element of a two-dimensional array, initially 0, and when all the contributions have been tallied, for each E and V the corresponding array element is divided by $2E$. This is the way I counted unrooted maps of genus G by number of edges and vertices. I did not attempt to derive an explicit formula, as Mednykh and Nedela did in [23]: the corresponding formula for G=0 in [28] already occupies half a page, and as G increases the formula would quickly become excessively long.

When I started on this project, Karabás hadn't yet debugged his program for calculating G-admissible orbifold signatures and the corresponding numbers of epimorphisms, and in any case I wanted to write a program to do those calculations myself, but I didn't use Harvey's condition for the existence of a G-admissible orbifold with a given signature and period or the Mednykh-Nedela formula (7) for one of the factors of the number of corresponding epimorphisms. Liskovets, perhaps wishing to do unto Mednykh and Nedela what they had done unto him (by generalizing to higher genus his use of quotient maps), refined both of these results [24], and I used Liskovets' refinement.

Here is a summary of the results I needed from [24]. Given an r-tuple $(m_1,\cdots,m_r)$ of integers, each greater than 1, let m be their least common multiple. For every prime p that divides m, let $a(p)$ be the exponent of p in the prime power factorization of m, and for each index j=1,2,…,r, let $a_j(p)$ be the corresponding exponent for m_j. Now let $s(p)$ be the number of indices j such that $a_j(p)=a(p)$. Then Harvey's condition is equivalent to the following:

E1: the Riemann-Hurwitz equation (8),

E2: m divides L,

E3: either $g\neq 0$ or $L\leq m$,

E4: $s(p) \neq 1$ for every odd prime p that divides m,

E5: if m is even, then $s(2)$ is also even.

The formula in [24] that replaces (7) is

$$E(m_1, \cdots, m_r) = \prod_{p \text{ prime}, p \text{ divides } m} (p-1)^{r(p)-s(p)+1} p^{v(p)} h_{s(p)}(p), \tag{14}$$

where $v(p) = \sum_{j=1,2,\cdots,r, a_j(p) \geq 1} \left(a_j(p) - 1 \right) - a + 1$, $r(p)$ is the number of m_j that are divisible by p, and $h_s(x) = \left((x-1)^{s-1} + (-1)^s \right) / x$. He also showed that $r \leq 2G+2$ and used the multiplicative formula for the k^{th} Jordan function

$$\phi_k(n) = n^k \prod_{p|m \text{ prime}} (1 - p^{-k}), \tag{15}$$

and revealed its important role in unlabeled enumeration (for diverse types of groups, maps and some other algebraic and topological objects).

I found (15) useful, especially for computing a table of $\phi_2(n)$, which is needed to compute unrooted toroidal maps, because such a table can be computed using a modified version of the sieve algorithm used to compute a table of $\phi(n)$: you initialize $\phi(n)$ to n^2 for every n up to the desired maximum B, and then for each p from 2 up to B such that $\phi(p)$ is still equal to p^2 (so that p is a prime), you let n run over all the multiples of p up to B and subtract $\phi(n)/p^2$ from $\phi(n)$. Also, since I used orbit lengths rather than branch indices to compute the number of vertices and faces in the original map that correspond to each vertex and face in the quotient map, I modified Liskovets' results to use the orbit length m_i/L instead of the branch index m_i, which simplifies the Riemann-Hurwitz equation (8).

All of my results agree with those of Giorgetti, Mednykh and Karabás, as well as with the number of unrooted maps with up to 11 edges that I generated (up to 6 edges in [32] and up to 11 edges in an as-yet-unpublished article) and the number of unrooted maps of genus G with $2G$ edges, one vertex and one face published in an article [33] that Liskovets was kind enough to bring to our attention. If any of you want source codes of my programs or tables I computed using them, you need only e-mail me at the address written just under the title of this article.

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Appendix. Number of unrooted maps with *E* edges and *V* vertices

E	V	genus 1	genus 2	genus 3	genus 4	genus 5
2	1	1				
2	sum	1				
3	1	3				
3	2	3				
3	sum	6				
4	1	11	4			
4	2	24				
4	3	11				
4	sum	46	4			
5	1	46	53			
5	2	180	53			
5	3	180				
5	4	46				
5	sum	452	106			
6	1	204	553	131		
6	2	1198	1276			
6	3	2048	553			
6	4	1198				
6	5	204				
6	sum	4852	2382	131		
7	1	878	4758	4079		
7	2	7212	18582	4079		
7	3	18396	18582			
7	4	18396	4758			
7	5	7212				
7	6	878				
7	sum	52972	46680	8158		

8	1	3799	35778	73282	14118	
8	2	40776	205867	167047		
8	3	142727	347558	73282		
8	4	212443	205867			
8	5	142727	35778			
8	6	40776				
8	7	3799				
8	sum	587047	830848	313611	14118	
9	1	16304	244246	970398	684723	
9	2	219520	1910756	3693031	684723	
9	3	999232	4747430	3693031		
9	4	2040348	4747430	970398		
9	5	2040348	1910756			
9	6	999232	244246			
9	7	219520				
9	8	16304				
9	sum	6550808	13804864	9326858	1369446	
10	1	69486	1552834	10556722	17586433	2976853
10	2	1139075	15680071	58591595	39630698	
10	3	6488604	52969260	97799324	17586433	
10	4	17227356	77948670	58591595		
10	5	23634214	52969260	10556722		
10	6	17227356	15680071			
10	7	6488604	1552834			
10	8	1139075				
10	9	69486				
10	sum	73483256	218353000	236095958	74803564	2976853
11	1	294350	9349284	99944546	319763792	195644427
11	2	5741220	117450580	748976684	1192082898	195644427
11	3	39779852	512308352	1823736772	1192082898	
11	4	132209016	1025303224	1823736772	319763792	
11	5	235876296	1025303224	748976684		
11	6	235876296	512308352	99944546		
11	7	132209016	117450580			
11	8	39779852	9349284			
11	9	5741220				
11	10	294350				
11	sum	827801468	3328822880	5345316004	3023693380	391288854