A Biased Survey of Map Enumeration Results

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Abstract: - This is a survey paper presenting map enumeration results obtained by the author, either alone or jointly with other researchers, as well as some of the other results in the literature that are the most relevant to the author's work. The results discussed in this paper include counting and generating maps and hypermaps of various types on orientable surfaces. The counting is achieved by means of closed-form formulas where possible, and otherwise - again, if possible - by computationally efficient algorithms.

***Key-Words:* - rooted maps; sensed maps; orientable surfaces; counting formulas; efficient algorithms**

1 Introduction

A *map* is a 2-cell imbedding of a connected undirected graph (loops and parallel edges allowed) in a closed surface. In this paper, the surface is assumed to be orientable and oriented unless otherwise indicated. If the surface is a sphere, then the map is called *planar*. The enumeration of planar maps was initiated by W.T. Tutte [1] as one of his many approaches towards solving the four-colour problem. This problem was eventually solved by other researchers by other means, but Tutte's work made an important contribution to the mathematical literature.

A map can be represented combinatorially as the imbedded graph together with a cyclic order imposed upon the *darts* (edge-vertex incidence pairs) at each vertex. Given a dart, the next dart in the same face is reached by first going to the other dart belonging to the same edge and then to the next dart in the cyclic order belonging to the same vertex. The genus g is found from the numbers n, m and f of vertices, edges and faces, respectively using the Euler-Poincaré formula:

$$n - m + f = 2(1-g). \tag{1}$$

This model was proposed by several researchers and its equivalence to the topological model was proved rigourously by G.A. Jones and D. Singerman [2]. An *isomorphism* between two combinatorial maps is an isomorphism between their imbedded graphs that preserves the cyclic order of the darts at each vertex and a *reversal* is a graph isomorphism that reverses this order; they correspond to an orientation-preserving homeomorphism and an orientation-reversing homeomorphism, respectively, between two topological maps. An *automorphism* of a map is an isomorphism between a map and itself with a similar definition for a reversal of a map. A *rooted map* is a map with a distinguished dart - its *root*. Only the trivial automorphism of a map preserves the root [3]; so rooted maps can be counted without considering their automorphisms. For enumeration purposes, a *sensed map* is an isomorphism class of maps and an *unsensed map* is an isomorphism-and-reversal class of maps; an *unrooted map* is either a sensed map or an unsensed map.

A planar map is called *2-connected* (or *non-separable*) if its underlying graph is 2-connected and has no loops; as an exception, the loop-map is considered 2-connected. A planar map is called *3-connected* if its underlying graph is 3-connected and has neither loops nor parallel edges.

A *hypermap* is a 2-cell imbedding of a hypergraph in an orientable surface. In the combinatorial model of a hypermap, a hyperedge can have an arbitrary number of darts and these too have a cyclic order imposed upon them.

The enumeration of rooted maps is treated in Section 2, that of sensed maps in Section 3 and that of unsensed maps in Section 4.

2 Rooted Maps

Tutte [3] obtained the following impressively simple closed-form formula for the number of rooted planar (1-connected) maps with m edges:

$$A'(m) = \frac{2 \times 3^m \times (2m)!}{m!(m+2)!}. \tag{2}$$

A similar closed-form formula [3] was found for rooted planar 2-connected maps. For rooted planar 3-

connected maps, the counting formula [3] contains a sum.

Enumeration formulas were later found for rooted planar 1-, 2- and 3-connected maps with $i+1$ vertices and $j+1$ faces. For 1-connected maps the formula, found by Tutte [4], is in the form of a parametric equation for the generating function. For 2-connected maps, the formula, found by W.G. Brown and Tutte [5], is in closed form. For 3-connected maps, the formula, found by R.C. Mullin and P.J. Schellenberg [6], contains a sum over several indices.

A.B. Lehman (my Ph.D. supervisor) and I found a closed-form formula for counting rooted planar loopless maps with m edges [7]

$$\frac{6(4m+1)!}{m!(3m+3)!} \tag{3}$$

and later I found one for rooted planar bipartite (or, dually, Eulerian) maps [8] with m edges

$$\frac{3 \times 2^{m-1} \times (2m)!}{m!(m+2)!} \tag{4}$$

as did other researchers independently. In [8] I also counted rooted planar bipartite maps by number of edges and faces. Rooted planar bipartite maps were counted by number of edges and number of vertices in each of the two parts by C. Chauve [9]. The formula is expressed as a system of parametric equations for the generating function.

Another pair of formulas, obtained jointly with Liskovets and L.M. Koganov [10], count the total number of faces in all the rooted planar loopless maps and in all the rooted planar bipartite maps with m edges

As part of my Ph.D. thesis I found a formula for the number of rooted one-face genus-g maps with m edges, a formula for the number of rooted maps with m edges without regard to genus and an exponential algorithm for counting rooted genus-g maps with m edges and n vertices [11]. As an intermediary result, this algorithm counts *slicings* - maps with specified vertex-degrees and a root in every vertex - by generalizing to higher genus Tutte's recursion for the number of planar slicings [1], for which he found simple closed-form formulas in the cases when all, or all but two, of the vertices are of even degree and when there are four vertices, all of odd degree. In another part of my thesis I presented another exponential algorithm [12] for counting rooted maps by genus, number of vertices and number of edges, which uses a code for rooted maps discovered by Lehman. Lehman never published his code; so I published it in an article [13] in which I used it to generate rooted maps of a given genus with a given number of vertices and number of edges as well as rooted planar maps of various types.

For *toroidal* (genus-1) rooted maps, I designed an algorithm which counts them with up to m edges by number of vertices and faces in $O(m^4)$ arithmetic operations [14]. Formulas for counting these maps by number of edges alone as well as by number of vertices and faces were later published by D. Arquès [15]. His formula for counting these maps by number of edges alone counts these maps with up to m edges in $O(m^2)$, which is more efficient than the algorithm in [14]; his formula for counting them by number of vertices and faces contains a sum over so many indices that it is less efficient than the algorithm in [14]. Independently, E.A. Bender, E.A. Canfield and Robinson [16] counted rooted maps with m edges on the torus and the projective plane.

Later Bender and Canfield counted rooted maps of genus 2 and 3 with m edges [17]. Independently, Arquès and A. Giorgetti counted rooted maps of arbitrary orientable genus by number of vertices and faces [18] and later did the same for rooted maps of arbitrary unorientable genus [19]. And no survey of map counting would be complete without mentioning that most prolific of authors on the subject, Liu Yanpei, who, with various co-authors, published numerous papers in which they counted rooted maps on the plane, the torus, the projective plane and the Klein bottle as a function of the degree of the face containing the root and the degree of the vertex containing the root as well as the number of edges. These results are collected in his book [20].

In [8] I presented a bijection between rooted hypermaps and rooted 2-coloured bipartite maps of the same genus. Given a hypermap H, its *incidence map* M is defined as follows: the white vertices of M are the vertices of H, the black vertices of M are the hyperedges of H, and a white vertex of M is adjacent to a black vertex of M if the corresponding vertex of H is incident to the corresponding edge of H. The edges of M correspond to the darts of H and the faces of M to the faces of H, except that the degree of each face of M is twice the degree of the corresponding face of H. The genus of a hypermap, defined by R. Cori [21], is obtained from the number n, m, d and f of vertices, hyperedges, darts and faces, respectively, by the formula

$$n + m - d + f = 2(1-g); \tag{5}$$

so the genus of M is equal to the genus of H. The incidence map of a bipartite map is often referred to in the literature as its Walsh map.

This correspondence is a bijection for rooted hypermaps and bipartite maps, although not for unrooted ones; I used it to count rooted planar hypermaps by number of darts and also by number of darts and faces. Chauve's formula for the number of rooted planar bipartite maps as a function of the number of black vertices, the number of white vertices and the number of edges yields a formula for the number of rooted planar hypermaps as a function of the number of vertices, edges and darts. A preliminary version of his paper contains a formula for this number with a sum over three indices. Arquès [22] also found parametric equations for the generating function for these numbers; his formula contains a sum over six indices. An interesting open problem is to obtain a formula that is computationally more efficient than a sum over three indices.

3 Sensed Maps

At first map enumeration was done only on rooted maps but, as shown in this section and the next one, once it became known how to treat map automorphisms and, later, reversals, many interesting results on the enumeration of unrooted maps began to appear in the literature.

The first sensed maps to be counted were sensed plane trees, done by D. Walkup [23]. Then V.A. Liskovets counted all sensed planar maps with m edges; the first publication containing this result is [24] and the one that contains the most general formulation of his method is [25]. His method uses the quotient map of a map with respect to a non-trivial automorphism. It is based on the observation that, given a planar map M and a non-trivial automorphism a of M of period p, M can be drawn on the sphere so that a is a rotation about an axis that intersects M in two *cells* (vertices, edges or faces). Such a geometrization of map symmetries is based on a theorem of P. Mani [26]. The map M is now expressible as p isomorphic copies of a smaller map Q, the *quotient map*, which occupies a lune that can be extracted and expanded into a sphere. Furthermore, M is uniquely determined by Q and a. Using the so-called Burnside's (orbit-counting) lemma [27], he reduced the enumeration of unsensed maps of a given type to the enumeration of rooted maps that are quotient maps of maps of that type. The quotient map of a planar map is just a planar map with two (arbitrary) axial cells distinguished; so the number of unsensed planar maps with m edges is given by

$$
\begin{aligned}
2mA^{+}(m) &= A'(m) \\
&+ \sum_{t \mid m, t<m} \phi\left(\frac{m}{t}\right)\binom{t+2}{2} A'(t) \\
&+ \begin{cases} \dfrac{m(m+3)}{2} A'\left(\dfrac{m-1}{2}\right) & \text{if } m \text{ is odd,} \\ \dfrac{m(m-1)}{2} A'\left(\dfrac{m-2}{2}\right) & \text{if } m \text{ is even,} \end{cases}
\end{aligned}
\tag{6}
$$

where $A'(m)$ is the the number of rooted planar maps with m edges given by (2) and ϕ is the Euler totient function.

Liskovets and I then collaborated to count (by number of edges alone) sensed planar maps of various types: non-separable [28], loopless [29], eulerian and bipartite [30], as well as all the above types of maps on the plane with a distinguished outside face [31]. All the counting formulas are of the same form as (6), which can be evaluated with up to m edges in $O(m \log m)$ operations.

Related results, obtained by M. Bousquet-Mélou and G. Shaeffer by using planar constellations, can be found in [32].

For 3-connected planar maps we were unable to characterize the quotient maps; this is still, to my knowledge, an open problem. To count sensed 3-connected planar maps I had to use other tools.

One such tool is the decomposition of a 2-connected graph into 3-connected components. There are several versions of this decomposition; the one I used is due to R.E. Tarjan and J. Hopcroft [33]. A related result, due to B.A. Trakhtenbrot [34], is a decomposition of a *2-pole network*, which is a 2-connected graph (parallel edges allowed but not loops) from which an edge has been removed and its incident vertices distinguished from each other and from the other vertices as the *poles*. Trakhtenbrot found a canonical decomposition of a 2-pole network into a core and components, where each edge of the core is replaced by a component. A statement of Trakhtenbrot's theorem in English appears in [35], where I used it to count labeled 3-connected graphs and 2-connected graphs with no vertices of degree 2. When this decomposition is repeated on all the components until they are reduced to single edges, the Hopcroft-Tarjan decomposition is obtained.

The other tool is the cycle index, which is a notation for describing the cycle structure of all the members of a permutation group. A formula for the cycle index of the Wreath product of two permutation groups was published by R.W. Robinson in [36] and used there, together with the decomposition of a connected graph into 2-connected components, to count unlabeled 2-connected graphs. In this decomposition, a connected graph is expressed as a 2-connected core together with the components, which are connected graphs with a distinguished vertex, each attached by its distinguished vertex to a vertex of the core; so only vertex cycles had to be included in the cycle index. In the Trakhtenbrot-Hopcroft-Tarjan decomposition, an edge of the core is replaced by a 2-pole network; so edge cycles had to be included as well. And since an edge of the core can have its orientation preserved or reversed when repeated applications of an automorphism take the edge into itself, preserving or exchanging the poles of the network that replaces the edge, two different types of edge cycles had to be included. F. Harary and E.M. Palmer [37] used these two types of edge cycles to count the directed graphs that can be made from a given undirected one. In [38] I extended Robinson's formula for the cycle index of a Wreath product to these two types of edge cycles and thereby counted unlabeled 3-connected graphs and 2-connected graphs with no vertices of degree 2.

The algorithms I used in [38] were so inefficient that I was able to count these graphs with up to only 9 vertices. Robinson then discovered a method of inverting equations in cycle index sums that enabled us to count unlabeled 2-connected graphs, 3-connected graphs and 2-connected graphs with no vertices of degree 2 with up to 18 vertices [39]. In [40] we attempted to express the cycle index of a Wreath product, extended to two types of edges cycles, in the language of species of structures, a religion among the combinatorialists in the Mathematics Department of UQAM. This task was later done rigourously by A. Gagarin, G. Labelle and P. Leroux [41], who called such a cycle index a Walsh cycle index.

In the case of sensed maps, the cycle index sum has so few terms that Robinson's inversion formula was unnecessary; so I was able to use the brute-force method of [38] to count sensed 3-connected planar maps and sensed 2-connected planar maps with no vertices of degree 2 by number of edges alone with up to *m* edges in $O(m^3)$ operations [42]. Later I added the number of vertices as a parameter [43]; once the number of rooted maps have been computed, to count sensed maps with up to *m* edges by number of vertices and edges takes $O(m^2)$ operations for 1- and 2-connected maps and $O(m^5)$ operations for 3-connected maps. This last time-complexity estimate was lowered to $O(m^2)$ by E. Fusy [44] by proving that all the necessary terms can be computed from the coefficients of algebraic generating functions, which can be extracted quickly.

Recently Robinson [45] counted sensed maps with *m* edges without regard to genus.

Using techniques of algebraic topology, A. Mednykh and R. Nedela generalized Liskovets' method to count sensed maps [46] and hypermaps [47] of genus 1, 2 and 3 by number of edges alone, using the formulas for rooted maps of these genera in [17]. Liskovets [48] later reformulated their method to make it more elegant and, although he didn't mention it, more efficient computationally as well. An interesting open problem is to write a computer program that uses the results of [46], [18] and [48] to count sensed maps by genus, number of vertices and number of edges.

4 Unsensed maps

The only enumeration of unsensed maps, aside from exhaustive search, of which I am aware was done by N.C. Wormald. In [49] and [50] he described an algorithm for counting unsensed 1-connected planar maps with up to *m* edges by number of edges and vertices in $O(m^6)$ operations, or $O(m^4)$ if one counts by number of edges alone. In addition, he graciously provided me with unpublished tables of numbers of sensed and unsensed 1-, 2- and 3-connected planar maps by number of edges alone and of sensed and unsensed 1- and 2-connected planar maps by number of edges and vertices. For sensed maps I was able to improve on these complexity estimates using Liskovets' quotient maps; for unsensed maps there are too many terms in the cycle indices to do so.

In [51] Liskovets characterized the quotient maps of unsensed planar maps under both automorphisms and reversals. An automorphism can be represented by a rotation of the embedding sphere and the quotient map is a map on the sphere. A reversal can be represented by a reflection in a plane passing through the centre of a sphere followed by a rotation about a diameter perpendicular to this plane. If the period *p* of the rotation is even, then the quotient map is a map on the projective plane. If *p* is odd (where *p*=1 means no rotation), then the quotient map is a map on the disk. Rooted maps on the projective plane have been counted by number of edges [16] and by number of edges and vertices [19]. But rooted maps on the disk have not yet been counted, although non-separable ones were counted by W.G. Brown [52], and to compute the cycle indices for the unsensed planar maps it would be necessary to count rooted maps on the disk by number of vertices, edges, vertices on the boundary, edges that intersect the boundary and edges that lie on the boundary; this is why the cycle indices have so many terms. Wormald computed his tables of numbers of unsensed 2- and 3-

connected planar maps without using cycle indices. Until he publishes his methods, the only way I know to count these maps is by generating them.

In [13] I generated not only rooted maps but also sensed and unsensed maps of a given genus with n vertices and m edges as well as planar maps of various types including 2-connected ones but not, unfortunately, 3-connected ones. These had already been generated by A.J.W. Duijvestijn and P.J. Federico [53]. Recently, G. Brinkmann and B. McKay [54] wrote a program to generate planar maps of various types including 3-connected ones by smart exhaustive search. I downloaded their program and, with the help of my colleague A. Gagarin, generated the 3-connected maps with up to 14 vertices, the highest number for which my computer has the necessary resources. I calculated and stored the cycle index sum for these maps using a computer program which I optimized by proving that for each automorphism or reversal of a planar map there are at most 3 lengths of vertex cycles and a total of 3 lengths between the two types of edge cycles (to be included in a future article).

Unsensed 3-connected planar maps (*polyhedra*) are particularly interesting because they link planar maps with planar graphs. A *planar graph* is a graph that can be imbedded in the plane. In general, a planar graph can be imbedded in the plane in several inequivalent ways. For example, consider a triangle ABC together with two other vertices, D and E, which are adjacent just to A. The same map on a sphere is obtained by putting both D and E inside the triangle or both of them outside of it, but if D is put inside the triangle and E outside of it, then we have a different map, which is not related to the other one by either isomorphism or reversal. However, it was proved by H. Whitney [55] that a 3-connected planar graph can be imbedded in the sphere in only one way up to isomorphism and reversal, from which it follows that any automorphism of the graph is either a map automorphism or a reversal of the imbedding, so that the cycle index sum for unsensed 3-connected planar maps is also the cycle index sum for unlabeled 3-connected planar graphs.

Using this cycle index sum, which I calculated for graphs with up to 14 vertices, and the equations of [38], I counted *strongly planar* 2-pole networks - that is, those which have no parallel edges and which are planar and remain planar when the poles are joined by an edge - and also strongly planar 2-pole networks that have an automorphism that exchanges their poles. The importance of these networks is that Gagarin, Labelle and Leroux expressed a 2-connected toroidal non-planar $K_{3,3}$-free graph as a composition whose core belongs to a family of graphs for each of which they computed the cycle index, and whose components are strongly planar 2-pole networks [41]. My work enabled Gagarin, Labelle and Leroux to extend their tables of numbers of these toroidal graphs [56].

Whitney's theorem, Trakhtenbrot's theorem and the Mullin-Shellenberg formula for counting rooted planar 3-connected maps were used by Bender, Zh. Gao and Wormald [57] to count labeled 2-connected planar graphs; this result was then used by M. Bodirsky, G. Gröpl and M. Kang [58] to count all labeled planar graphs. From the cycle index sum for 3-connected planar graphs it is possible to count unlabeled planar graphs - first 2-connected ones, then connected ones and then all of them (to be included in the above-mentioned future article). Once a method has been discovered for computing this cycle index sum more efficiently than by exhaustive search, it will be possible to count unlabeled planar graphs (a problem posed by Liskovets and me [59]) with more than 14 vertices and to extend the tables of Gagarin, Labelle and Leroux still further. Perhaps unlabeled planar graphs - and the toroidal graphs considered by Gagarin, Labelle and Leroux - can be counted without using cycle indices, as Wormald did for unsensed 2- and 3-connected maps. A solution to these problems using either approach would be of great interest.

Anyone wishing to look at the tables of numbers I have computed can find them in either the articles in the following bibliography or in the 120 contributions I have made, some of them jointly with other researchers, to N.J.A. Sloane's data base of integer sequences (OEIS) [60].

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