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THE MONADIC THEORY OF MODULAR DECOMPOSITION

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Abstract. We define (by a Monadic Second Order Logic formula) in any 2-structure another 2-structure having the same family of modules. While already the first-order theory of 2-structures is undecidable, we show that the Monadic Second Order theory of these defined structures is decidable by automata on trees. We also give an explicit description of the constructed 2-structures and a recurrence to compute its number of equivalence classes. Finally we show that similar results hold for symmetric 2-structures.

1. Introduction

A *module* (see Section 2 for formal definitions) in a graph is a set of vertices which are indistinguishable from the outside. This means that if there is an edge (or an arc in the oriented case) between some vertex v not in the module and a vertex in the module, then there is an edge (or an arc, of the same orientation), between v and any other element of the module. The empty set, singletons and the set of all vertices are obviously modules; they are called the *trivial modules*.

For A and B two disjoint modules, there are either an edge (arc) between any element of A and any element of B or no edge (arc) between any element of A and any element of B. Therefore a partition of the set of vertices of a graph into a disjoint union of modules gives a natural quotient structure in which each module of the partition is collapsed to a single vertex. Furthermore it has been shown [1, Chapter 5] that the partition into maximal strong modules (called maximal prime clans in [1]) can only give three kinds of quotients. Reiterating this operation on the subgraphs induced on the maximal strong modules gives a tree which is called the *modular decomposition tree*.

The modular decomposition has been used for designing efficient algorithms for classes of graphs (see for instance [2]). It has also been used to draw graphs [3, 4]. Finally the design of efficient algorithms for the modular decomposition has been a quite active area of research. It has been shown that the modular decomposition both for graphs and 2-structures can be computed in time linear in the number of nodes and edges [5–8].

The definition of module impose uniformity on the type of relation holding between an element outside it and all elements inside, but says nothing about the exact type of relation. For instance, a non-oriented graph has the same set of modules as its complement (edges and non-edges interchanged). Similarly, an oriented graph has the same modules as its reverse graph (arcs' orientations reversed). In fact, to define a module one need only to distinguish between the type of relations holding between oriented pairs of vertices. This fact was noticed by [9, 10] who

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introduced the notion of 2-*structure* as a finite set with an equivalence relation on the set of distinct ordered pairs.

As we just said, both directed and undirected graphs can be considered as 2-structures. An undirected graph being a 2-structure whose equivalence relation has two classes: one for the edges and one for the non-edges. Similarly a directed graph is a 2-structure whose equivalence relation has four classes: one class for the arcs in one direction, one class for the arcs in the other direction, one for pairs of vertices linked by arcs in both directions and finally one class for pairs of vertices linked by no arc. The notion of 2-structure is hence a natural setting in which to develop the *modular decomposition* of graphs [11–17].

Furthermore modular decomposition can be defined set-theoretically. Indeed, a *module* is a set of elements of a 2-structure defined in terms of the equivalence relation. It has been shown by [9, 10, 18–20] that the family of modules form a so called *weak partitive set family* (WPSF). Furthermore, [1, Theorem 5.7] has shown that every weak partitive set family is the family of modules of some 2-structure, which shows that the notion of 2-structure is *complete* (in the logical sense) for the set-theoretical axiomatization of WPSF (Section 2, Definition 3). Note that there are WPSF which are not the family of modules of any graph (oriented or not) (see Section 10), so 2-structures form indeed a very natural setting in which to cast modular decomposition.

The objective of this paper is to revisit this completeness result from a logical point of view. The emphasis will be on the amount of information on the 2-structure's equivalence relation which can be recovered from the WPSF alone.

The notion of 2-structure is first-order definable since we just have to state that the relation is an equivalence relation on distinct pairs. Nevertheless the notion of module, which is a set of elements, is not first-order. As one can easily show (see Section 5), the set of modules of a 2-structure is definable by a *Monadic Second Order* (MSO) formula $MODULES(X)$ in one monadic variable X.

The main results of this paper are the following. For a WPSF $\mathcal{F}$ on a set S take $\rightleftharpoons_{\mathcal{F}}$ to be the intersection of all equivalence relations $\equiv$ such that the family of modules of the 2-structure $\langle S, \equiv\rangle$ is $\mathcal{F}$. Let MSO-2SF denotes the MSO-theory of the structures $\langle S; \rightleftharpoons_{\mathcal{F}}\rangle$ where $\mathcal{F}$ is a WPSF on S (the set of MSO sentences true in all these structures). We will show that:

(1) The family of modules of $\langle S, \rightleftharpoons_{\mathcal{F}}\rangle$ is $\mathcal{F}$.
(2) The relation $\rightleftharpoons_{\mathcal{F}}$ is MSO-definable in any 2-structure whose set of modules is $\mathcal{F}$.
(3) The class of models $\mathcal{M}$ of MSO-2SF is MSO-definable, which means that there is an MSO sentence φ such that a 2-structure satisfies φ if and only if it is in $\mathcal{M}$.
(4) MSO-2SF is decidable by interpretation in the Weak Monadic Second Order Theory of 2 successors (WMS2S).

These facts can be interpreted as follows. Take $\langle S, \equiv\rangle$ to be any 2-structure and $\mathcal{F}$ its family of modules. The first fact gives that $\langle S, \equiv\rangle$ and $\langle S, \rightleftharpoons_{\mathcal{F}}\rangle$ have the same family of modules. The second fact shows that the MSO theory of $\langle S, \rightleftharpoons_{\mathcal{F}}\rangle$ is the "minimal" part of the MSO theory of $\langle S, \equiv\rangle$ needed to recover the modules of $\langle S, \equiv\rangle$. Indeed our construction will show that $\rightleftharpoons_{\mathcal{F}}$ is definable from $\mathcal{F}$ alone, so as long as $\mathcal{F}$ is MSO-definable in a structure $\mathcal{M}$ on the set S, we can define $\rightleftharpoons_{\mathcal{F}}$ in $\mathcal{M}$

and hence interpret the whole MSO theory of $\langle S, \rightleftharpoons_{\mathcal{F}} \rangle$ in $\mathcal{M}$. The third fact gives that MSO-2SF is some sort of "natural", since it is equivalent to an MSO-sentence.

Furthermore, the *first-order theory of finite 2-structures* is undecidable by the well-known result of Trakhtenbrot [21] (see Section 5). Therefore this is also the case for the MSO-theory of finite 2-structures, which we will denote by MSO-2S.

The last fact (4) above shows that if we restrict our attention to modules and modular decompositions and consider MSO-2SF, this last theory is interpretable in WMS2S and is therefore, by the well known result of [22,23], decidable by automata on trees.

As we said, there are WPSF which are not families of modules of graphs. In fact, it is known (see section 10) that any WPSF is the family of modules of a 2-structure with 6 equivalences classes, but in general not less. One may therefore wonder how many classes $\rightleftharpoons_{\mathcal{F}}$ may have. Furthermore $\langle S, \rightleftharpoons_{\mathcal{F}} \rangle$ being a combinatorial structure, one would expect an explicit description. We will hence give in section 10 an explicit description of $\rightleftharpoons_{\mathcal{F}}$ and furthermore show how to compute its number of equivalence classes from the modular decomposition tree.

A non-oriented graph can be seen as an oriented graph such that if there is an arc from vertex v to w then there is also a returning edge from w to v. Similarly a symmetric 2-structure is a 2-structure such that the distinct pairs (x, y) and (y, x) are always in the same equivalence class. The set of modules of a symmetric 2-structure is a family of sets called a *partitive set family* (PSF). We will finally show that results, similar to the above, hold for PSF.

The Monadic Second Order theory of graphs has been extensively studied (see [24] for a survey). In particular, the modular decomposition of a graph (as a tree) is MSO-definable in a graph structure if one uses an arbitrary linear order on the vertices [25] (see also [26]).

The point of view in this paper is quite different. As we said, a WPSF (or a PSF in the symmetric case) is enough to compute the modular decomposition. In this work we investigate the concept of WPSF (and PSF in the symmetric case) and ask which kind of information on the equivalence relation can be obtained from the set family alone. This allows us to exhibit a class (the models of MSO-2SF) of 2-structures whose families of modules cover all WPSF's. Furthermore its theory, MSO-2SF, being decidable by automata is more tractable that even the first-order theory of 2-structures.

This paper is structured as follow. Section 2 introduces terminology and gives a criterion for membership in a WPSF, Section 3 gives a construction for $\rightleftharpoons_{\mathcal{F}}$ and shows that it is included in any equivalence relation $\equiv$ such that the family of modules of $\langle S, \equiv \rangle$ is $\mathcal{F}$, Section 4 shows that the family of modules of $\langle S, \rightleftharpoons_{\mathcal{F}} \rangle$ is indeed $\mathcal{F}$, Section 5 shows MSO-definability of $\rightleftharpoons_{\mathcal{F}}$ and MSO-2SF, Section 6 presents a first-order analogue of MSO-2SF and Section 7 recalls facts about the modular decomposition which we will use in Section 8 to give our interpretation in WMS2S. In section 9 we turn our attention to *labeled 2-structures* in order to give in section 10 a construction from its family of modules of a 2-structure, whose equivalence relation has 6 classes. We will also give in that section a direct construction of $\rightleftharpoons_{\mathcal{F}}$, computing its number of classes from the modular decomposition tree. In section 11 we show how similar results hold for PSF's. Finally Section 12 concludes the paper.

2. 2-Structures, Modular Set Families and Weak Partitive Set Families

Let us settle some terminology and notation. We will say that a set S *avoids* an element x if x is not in S. Two sets *intersect* if their intersection is non-empty. The *difference* $X \setminus Y$ of two sets X, Y, is the set of all elements of X which are not in Y. For S a set, a *distinct pair* of S is an ordered pair (x, y) of elements of S such that $x \neq y$. All the sets and structures considered in this paper are finite.

Definition 1 (2-Structure)**.** *A* 2-structure *is a set B equipped with an equivalence relation $\equiv$ on the set of distinct pairs of B. It will be denoted by $\langle B; \equiv \rangle$.*

A *module* of a 2-structure $\langle B; \equiv \rangle$ is a subset M of B which "looks the same" from any point outside it, more formally we have the following definition.

Definition 2 (Module)**.** *Let $\mathcal{B} = \langle B; \equiv \rangle$ be a 2-structure. A* module *of $\mathcal{B}$ is a subset M of B such that, for all $x \notin M$, $y, z \in M$, $(x, y) \equiv (x, z)$ and $(y, x) \equiv (z, x)$.*

We will denote by $Modules(\mathcal{B})$ the family of modules of the 2-structure $\mathcal{B}$. We will also say that a family of sets is a *modular set family* if it is the family of modules of some 2-structure. Modular set families can be axiomatized by the following notion, which is called a *siba (semi-independent boolean algebra)* in [1]. This gives a setting in terms of set operations appropriate for the development of the modular decomposition of 2-structures (see [1]).

Let us say that two sets X and Y *overlap* if $X \setminus Y$, $X \cap Y$, $Y \setminus X$ are all non-empty.

Definition 3 (Weak Partitive Set Families)**.** *A* weak partitive set family *(WPSF) on some set S is a family of subsets $\mathcal{F}$ of S such that*

1. *S, $\emptyset$ and all singletons are in $\mathcal{F}$.*
2. *$X \cap Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$.*
3. *$X \cup Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$ and overlap.*
4. *$X \setminus Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$ and overlap.*

Note that if X, $Y \in \mathcal{F}$ do not overlap then the intersection is either $\emptyset$, X or Y and is hence trivially in $\mathcal{F}$. Therefore the overlapping condition could also be added to the second case. Note also that if two sets X, Y intersect without overlapping, then $X \cup Y$ is either X or Y. So in the third case the condition could be replace by "intersect" instead of "overlap".

Furthermore, one easily shows [1, Lemma 3.5] that any modular set family is a WPSF. Conversely, [1, Theorem 5.7] shows by induction, that every weak partitive set family is the family of modules of some 2-structure. We will need a constructive proof (Corollary 1) of this result; giving an equivalence relation which is MSO-definable from a WPSF. But in order to show that the set of modules of the constructed equivalence relation is exactly the WPSF, we will first need the following characterization from [20].

Proposition 1 ([20, Lemma 2.1.1])**.** *Let $\mathcal{F}$ be a WPSF on S. For X a subset of S the two following conditions are equivalent.*

1. *X is in $\mathcal{F}$*
2. *For all $x \notin X$ and all $y, z \in X$ there exists an $F \in \mathcal{F}$, such that $x \notin F$, $y, z \in F$.*

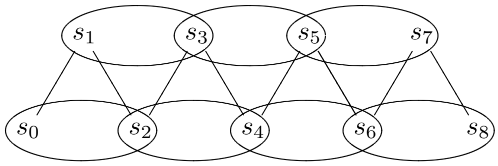

FIGURE 1. A Zigzag

proof. If *(1)* holds then X itself fulfills the condition on F of *(2)*. This shows that *(1)* implies *(2)*.

For the converse, consider an X satisfying statement *(2)*. If X is empty then by definition $X \in \mathcal{F}$. Otherwise we will show that there is for any $x \notin X$ an $F_x \in \mathcal{F}$ such that $x \notin F_x$ and $X \subseteq F_x$. This will give that $\bigcap_{x \in X} F_x$, which is in $\mathcal{F}$, contains X and avoids every element of its complement. Hence $\bigcap_{x \in X} F_x = X$, showing that X is indeed in $\mathcal{F}$.

Consider a non-empty X and an $x \notin X$. Taking $y = z$ to be any element of X we get by *(2)* that there is an element of $\mathcal{F}$ which avoids x but intersects X. Let now F_x be the greatest (for inclusion) element of $\mathcal{F}$ which avoids x and intersects X. We will now show that F_x contains X.

Consider a y of X. Taking z to be any element of $F_x \cap X$ we get from *(2)* an $F \in \mathcal{F}$ containing y and z but not x. Now $F_x \cup F \in \mathcal{F}$ since z is in F_x and F. By maximality of F_x, the sets F_x and $F_x \cup F$ are equal, hence $y \in F_x$. Since this holds for any $y \in X$, it follows that F_x contains X. □

3. ZIGZAGS

Let $\mathcal{B} = \langle B; \equiv \rangle$ be a 2-structure and $s_0, s_1, s_2 \in B$. If $s_0, s_2 \in M$ and $s_1 \notin M$ for some module M of $\mathcal{B}$ (see Figure 1), then by the definition of module we have that $(s_0, s_1) \equiv (s_2, s_1)$. Iterating this construction we can recover some information on $\equiv$ from a WPSF. We hence define the following notion.

Definition 4 (zigzag)**.** *Let $\mathcal{F}$ be a WPSF on the set S. An $\mathcal{F}$-*zigzag *between distinct pairs (a, b) and (c, d) of S is a sequence $s_0, \ldots, s_n$, $n \geq 1$ (see Figure 1) such that*

1. *$a = s_0$, $b = s_1$.*
2. *$s_{n-1} = c$ and $s_n = d$ if n is odd and $s_{n-1} = d$ and $s_n = c$ if n is even.*
3. *for all $i = 0, \ldots, n-2$ there exists an $F \in \mathcal{F}$ such that $s_i, s_{i+2} \in F$ and $s_{i+1} \notin F$.*

We will say that n is the length *of the zigzag. Furthermore, we will denote the existence of an $\mathcal{F}$-zigzag between (a, b) and (c, d) by $(a, b) \rightleftharpoons_{\mathcal{F}} (c, d)$.*

Remark 1. *Consider $s_0, \ldots, s_n$ an $\mathcal{F}$-zigzag between (a, b) and (c, d). The sequence $s_0, \ldots, s_n, s_{n-1}$ obtained from $s_0, \ldots, s_n$ by repeating the second last element s_{n-1} at the end is also an $\mathcal{F}$-zigzag between (a, b) and (c, d) (since $\{s_{n-1}\}$ contains s_{n-1} and excludes s_n). Therefore one can consider without loss of generality that an $\mathcal{F}$-zigzag is of an even length. This fact will be convenient to simplify the presentation of some of our proofs.*

Note also that if $s_0, \ldots, s_n$ *is an* $\mathcal{F}$*-zigzag between* (a, b) *and* (c, d)*, then adding* s_1 *at the beginning and* s_{n-1} *at the end of this sequence gives* $s_1, s_0, \ldots, s_n, s_{n-1}$ *which is an* $\mathcal{F}$*-zigzag between* (b, a) *and* (d, c)*. We therefore have that* $(a, b) \rightleftharpoons_{\mathcal{F}} (c, d)$ *if and only if* $(b, a) \rightleftharpoons_{\mathcal{F}} (d, c)$ *(the relation* $\rightleftharpoons_{\mathcal{F}}$ *is said to be* reversible*).*

The notion of zigzag is reminiscent of a construction used by Gallai [11, 12] to study orientations of undirected graphs. However Gallai considered sequences of vertices $s_0, \ldots, s_n$ of an undirected graph such that $s_i s_{i+1}$, $i = 0, \ldots, n-1$ are edges and $s_i s_{i+2}$, $i = 0, \ldots, n-2$ are non-edges, while zigzags are defined only in terms of the WPSF. As a consequence [11, 12, Theorem 3.1.5] shows that if two consecutive vertices s_i, s_{i+1} are in a module M then all s_i, $i = 0, \ldots, n$ are in M. Such a result obviously doesn't hold in our case. Consider for instance the graph with no edge. Any sequence $s_0, \ldots, s_n$ of vertices of this graph forms a zigzag, while $\{s_i, s_{i+1}\}$ is a module containing only two vertices of the sequence $s_0, \ldots, s_n$. Furthermore Lemma 2 below obviously doesn't hold in Gallai's setting. These two notions are hence distinct.

In order to be a tentative relation defining a 2-structure, $\rightleftharpoons_{\mathcal{F}}$ must first be an equivalence relation.

Proposition 2. *Let* $\mathcal{F}$ *be a WPSF on the set* S*. The relation* $\rightleftharpoons_{\mathcal{F}}$ *is an equivalence relation on the set of distinct pairs of* S*.*

proof. We must show that $\rightleftharpoons_{\mathcal{F}}$ is reflexive, symmetric and transitive.

For $a, b \in S$, the sequence a, b, a is an $\mathcal{F}$-zigzag between (a, b) and itself since $\{a\}$ is an element of $\mathcal{F}$ which contains a and avoids b. Hence $\rightleftharpoons_{\mathcal{F}}$ is reflexive.

To show that $\rightleftharpoons_{\mathcal{F}}$ is symmetric, consider $s_0, \ldots, s_n$ an $\mathcal{F}$-zigzag between (a, b) and (c, d). By the previous remark, we can assume that n is even. Now $s_n, \ldots, s_0$ is also an $\mathcal{F}$-zigzag between (c, d) and (a, b), showing that $\rightleftharpoons_{\mathcal{F}}$ is indeed symmetric.

For transitivity, consider $a, b, c, d, e, f \in S$, such that $(a, b) \rightleftharpoons_{\mathcal{F}} (c, d)$ and $(c, d) \rightleftharpoons_{\mathcal{F}} (e, f)$. Let furthermore $s_0, \ldots, s_n$ be the $\mathcal{F}$-zigzag between (a, b) and (c, d) and $t_0, \ldots, t_m$ be the $\mathcal{F}$-zigzag between (c, d) and (e, f). We can again assume that n is even, so $s_0, \ldots, s_{n-1} = d, s_n = c = t_0, t_1 = d, \ldots, t_m$ is an $\mathcal{F}$-zigzag since $\{d\} \in \mathcal{F}$ contains d and avoids c. So $(a, b) \rightleftharpoons_{\mathcal{F}} (e, f)$, showing the claim. □

Different 2-structures on the same underlying set can have the same family of modules. For instance all 2-structures on a two element set have the same family of modules since there is only one WPSF on this set (the power set). But with two distinct pairs there are two equivalence relations, one with a single equivalence class and one with two classes (see also section 9 for a more elaborate example). So one cannot hope that $\rightleftharpoons_{\mathcal{F}}$ would always be the original equivalence relation. Nevertheless we have the following inclusion.

Proposition 3. *Let* $\mathcal{F}$ *be the family of modules of some* 2*-structure* $\langle B; \equiv \rangle$*. The relation* $\rightleftharpoons_{\mathcal{F}}$ *is finer than* $\equiv$ *(i.e.* $\rightleftharpoons_{\mathcal{F}} \subseteq \equiv$*).*

proof. It is sufficient to show by induction on $i = 0, \ldots n-1$ that for an $\mathcal{F}$-zigzag $s_0, \ldots, s_n$ the following holds.

$$\begin{aligned} (s_0, s_1) &\equiv (s_i, s_{i+1}) \text{ if } i \text{ is even} \\ (s_0, s_1) &\equiv (s_{i+1}, s_i) \text{ if } i \text{ is odd} \end{aligned}$$

The case $i = 0$ being clear, consider an i satisfying these properties.

Table 1. Conditions on the Sets of Lemma 1

$b \notin A$	$b, d \in B$	$d \notin C$	$d, f \in D$	$f \notin E$
$a, c \in A$	$c \notin B$	$c, e \in C$	$e \notin D$	$e, g \in E$

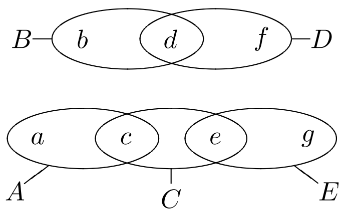

Figure 2. The Sets of Lemma 1

If i is even then $(s_0, s_1) \equiv (s_i, s_{i+1})$ and $i + 1$ is odd. Furthermore by the definition of zigzags there exists an $F \in \mathcal{F}$ containing s_i, s_{i+2} avoiding s_{i+1}. Since $\mathcal{F}$ is the family of modules of $\langle B; \equiv\rangle$, it follows that $(s_i, s_{i+1}) \equiv (s_{i+2}, s_{i+1})$ showing that the above properties also holds for $i + 1$.

The case of i odd is similar. □

Definition 5 (Realization)**.** *Let $\mathcal{F}$ be some WPSF on the set S. We will say that a binary equivalence relation $\equiv$ on distinct pairs of S* realizes *$\mathcal{F}$ if $\mathcal{F} = Modules(\langle S; \equiv\rangle)$. In such a case we will also say that $\langle S; \equiv\rangle$* realizes *$\mathcal{F}$*

By the last proposition, for any 2-structure $\langle B; \equiv\rangle$ which realizes $\mathcal{F}$ we have that $\rightleftharpoons_{\mathcal{F}}$ is finer that $\equiv$. If we furthermore had that $\rightleftharpoons_{\mathcal{F}}$ realizes $\mathcal{F}$, then $\rightleftharpoons_{\mathcal{F}}$ would be the finer such relation and hence it would be the intersection of all binary equivalence relations realizing $\mathcal{F}$. We will now show that this is indeed the case.

4. Zigzags' Realizations

We will now prove that for a WPSF $\mathcal{F}$, the modules of $\langle S; \rightleftharpoons_{\mathcal{F}}\rangle$ are exactly the elements of $\mathcal{F}$.

Theorem 1. *Let $\mathcal{F}$ be a WPSF on S. The family of modules of $\langle S; \rightleftharpoons_{\mathcal{F}}\rangle$ is equal to $\mathcal{F}$. Furthermore $\rightleftharpoons_{\mathcal{F}}$ is the intersection of all equivalence relations which realize $\mathcal{F}$.*

Corollary 1. *Any WPSF is a modular set family.*

In order to prove Theorem 1, we will need the following lemmas. Lemma 1 below is our main technical result. It shows that one can replace a length 6 zigzag by a length 4 one.

Lemma 1. *Let $\mathcal{F}$ be a WPSF on S. Let also A, B, C, D, E be elements of $\mathcal{F}$ (see Figure 2) satisfying the conditions of Table 1. There exists $X, Y, Z \in \mathcal{F}$, such that $a \in X$, $b \notin X$, $b, f \in Y$, $g \in Z$, $f \notin Z$ and $X \cap Z$ is not included in Y.*

proof. We consider three cases.
Case 1: If $b \notin C$, we will show that $X = A \cup C$, $Y = (B \setminus C) \cup D)$, $Z = E$ fulfill the statement of the Lemma. First, A and C have a non-trivial intersection which

contains c, so $X \in \mathcal{F}$. Secondly, C contains c which is not in B, hence $C \setminus B$ is non-empty and $B \setminus C$ is in $\mathcal{F}$. Now d is both in $B \setminus C$ and D, so $Y \in \mathcal{F}$.

Secondly, since a is in A, it follows that it is also in X. From $b \notin C$ it follows that $b \in Y$. Similarly, f being in D is also in Y. It remains to be shown that $X \cap Z$ is not included in Y. But e, which is in C and E, is also in $X \cap Z$, but it is neither in $B \setminus C$ (since it is in C) nor in D (by hypothesis), completing this case.
Case 2: Symmetrically (flip Figure 2 along a vertical axis) if $f \notin C$ then $X = A$, $Y = (D \setminus C) \cup B)$, $Z = C \cup E$ fulfill the statement of the Lemma.
Case 3: Finally, let $b, f \in C$. If $c \notin D$ then we can replace C by $C \setminus D$ and still fulfill the hypothesis of the Lemma, but this time the case $f \notin C$ above applies, completing the proof. Symmetrically, if $e \notin B$ then we can replace C by $C \setminus B$ and still fulfill the hypothesis of the Lemma, but this time the case $b \notin C$ above applies, completing this case also.

We are hence left with the case $b, f \in C$, $c \in D$ and $e \in B$. We will now show that we can then choose $X = A \cup (D \setminus (B \cap C))$, $Y = C$ and $Z = E \cup (B \setminus (D \cap C))$.

First $B \cap C$ contains e which is not in D, hence $(B \cap C) \setminus D$ is non-empty and therefore $D \setminus (B \cap C)$ is in $\mathcal{F}$. Now c is in D but not in B, therefore c is neither in $B \cap C$. It follows that c is in $D \setminus (B \cap C)$ and since it is in A, we have that X is in $\mathcal{F}$. A symmetrical argument gives that $Z = E \cup (B \setminus (D \cap C))$ is also in $\mathcal{F}$.

One easily checks that $a \in X$, $b \notin X$, $b, f \in Y$, $g \in Z$ and $f \notin Z$, so it remains to be shown that $X \cap Z$ is not contained in Y. Note that d is in D but not in C, hence it is in $D \setminus (B \cap C)$. Symmetrically d is in $B \setminus (D \cap C)$. We hence have that $d \in X \cap Z$. Since by hypothesis d is not in C, the proof is completed. □

We can now show that in fact any zigzag can be replace by a length 4 one.

Lemma 2. *Let $\mathcal{F}$ be a WPSF on S. If $(r, s) \rightleftharpoons_{\mathcal{F}} (u, v)$, then there exists a length 4 $\mathcal{F}$-zigzag between (r, s) and (u, v).*

proof. As we said in Remark 1, if there is an $\mathcal{F}$-zigzag between (r, s) and (u, v), there is one of an even length. Consider the shortest $\mathcal{F}$-zigzag of even length between (r, s) and (u, v). If its length is greater than 4 take a, b, c, d, e, f, g to be its last 7 elements. These 7 elements satisfy the hypothesis of Lemma 1, hence there exists $X, Y, Z \in \mathcal{F}$, such that $a \in X$, $b \notin X$, $b, f \in Y$, $g \in Z$, $f \notin Z$ and $X \cap Z$ is not included in Y. Take h to be an element of $(X \cap Z) \setminus Y$. If we replace the last 7 elements a, b, c, d, e, f, g of our $\mathcal{F}$-zigzag by a, b, h, f, g we get another $\mathcal{F}$-zigzag of a shorter length. Therefore the shortest $\mathcal{F}$-zigzag must be of length at most 4.

Finally if the $\mathcal{F}$-zigzag is of length smaller than 4, we can by the method of Remark 1 increase its length to 4. We therefore always get an $\mathcal{F}$-zigzag of that length. □

Example 1. *Let us show that the previous result is as sharp as possible and that there are some zigzags which cannot be replaced by a length 3 one.*

Consider $S = \{a, b, c, d, e\}$ and the WPSF $\mathcal{F}$ on S formed of the empty set, the singletons, S and of the following sets $\{a, c\}$, $\{a, d\}$, $\{b, e\}$, $\{c, d\}$, $\{a, c, d\}$, $\{a, b, e\}$, $\{b, d, e\}$, $\{b, c, e\}$, $\{b, c, d, e\}$, $\{a, b, c, e\}$, $\{a, b, d, e\}$.

One can easily check that this set family is indeed a WPSF. Furthermore a, b, c, d, e is a length 4 zigzag between (a, b) and (e, d). Finally there is no length 3 zigzag between these pairs since a, b, e, d is no zigzag, which follows from the fact that any set of $\mathcal{F}$ containing b and d also contains e.

We are now ready for the proof of the Theorem 1.

proof. **(of Theorem 1)** We must show that $X \in \mathcal{F}$ if and only if X is a module of $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$.

Let us first show that if $X \in \mathcal{F}$ then X is a module of $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$. Consider $x \notin X$ and $y, z \in X$. Since $X \in \mathcal{F}$ it follows that x, y, x, z, x and y, x, z are $\mathcal{F}$-zigzags. We therefore have that $(x, y) \rightleftharpoons_{\mathcal{F}} (x, z)$ and $(y, x) \rightleftharpoons_{\mathcal{F}} (z, x)$ respectively and X is therefore a module of $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$.

Conversely let us now show that if X is a module of $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$, then $X \in \mathcal{F}$. Using Proposition 1 it is sufficient to find for any $x \notin X$ and $y, z \in X$ an $F_x \in \mathcal{F}$ such that $x \notin F_x$ and $y, z \in F_x$.

Now since X is a module of $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$, we have that for any $x \notin X$, $y, z \in X$, $(x, y) \rightleftharpoons_{\mathcal{F}} (x, z)$. By Lemma 2, there is a length 4 $\mathcal{F}$-zigzag, x, y, t, z, x between (x, y) and (x, z). Let $X \in \mathcal{F}$ be a set which contains x, t and avoids y, $Y \in \mathcal{F}$ be a set which contains y, z and avoids t and finally $Z \in \mathcal{F}$ be a set which contains t, x and avoids z. Now $Y \setminus (X \cap Z) \in \mathcal{F}$, since $t \in X \cap Z$. It now suffices to take F_x to be $Y \setminus (X \cap Z)$. □

5. The logic of 2-structures

As we said in the introduction, the notion of 2-structure is definable in the first-order language of $\{\equiv\}$ since we just have to state that $\equiv$ is an equivalence relation on distinct pairs. Note that here $(x, y) \equiv (u, v)$ is a relation on the four-tuple (x, y, u, v) and that we always consider equality to be part of the language since we need it to define the notion of distinct pair.

However, the *first-order theory of finite* 2*-structures* which is the set FO-2S of first-order sentences true in every 2-structure is undecidable. This can easily be shown using the well-known result of Trakhtenbrot [21] which shows that the first-order theory of finite structures with one binary relation is undecidable.

Indeed, we will now show that there is an interpretation of the first-order theory of one binary relation FO-1BR in FO-2S, which means that for any sentence φ in the language of one binary relation R, there is a sentence $\tilde{\varphi}$ in the language $\{\equiv\}$ such that $\varphi \in$FO-1BR if and only if $\tilde{\varphi} \in$FO-2S.

First define for a sentence φ in the language of one binary relation R the sentence $\hat{\varphi}(x, y, z)$ in three new variables x, y, z (we assume that these new variables don't appear in φ) in the following way. $\hat{\varphi}(x, y, z)$ is obtained from φ by replacing $R(u, v)$ by $u \neq v \wedge (u, v) \equiv (x, y) \vee u = v \wedge (u, z) \equiv (x, y)$ and each quantifier Qt by $Qt \neq z$. Now $\tilde{\varphi}$ is $\forall xyz(x \neq y \rightarrow \hat{\varphi}(x, y, z))$. We will now show that this is indeed an interpretation.

For a structure $\mathcal{S} = \langle S; R \rangle$, such that S is a finite set and R a binary relation, consider the 2-structure $\mathcal{S}_{abc} = \langle S \cup \{c\}; \equiv \rangle$, where c is a new element not present in S, while a, b are some distinct elements of S. Define $(u, v) \equiv (a, b)$ if $R(u, v)$ and u, v are distinct elements of S and $(u, c) \equiv (a, b)$ if $R(u, u)$ for u an element of S. It now follows that $\mathcal{S} \models \varphi$ if and only if $\mathcal{S}_{abc} \models \hat{\varphi}(a, b, c)$. Therefore if $\varphi \notin$FO-1BR then $\tilde{\varphi} \notin$FO-2S.

Conversely consider a 2-structure $\mathcal{B} = \langle S; \equiv \rangle$ and let a, b, c be elements of S such that $a \neq b$. We can define the structure $\mathcal{B}_{abc} = \langle S \setminus \{c\}; R \rangle$ where $R(u, v)$ holds if either $u \neq v$ and $(u, v) \equiv (a, b)$ or $u = v$ and $(u, c) \equiv (a, b)$. It now follows that $\mathcal{B}_{abc} \models \varphi$ if and only if $\mathcal{B} \models \hat{\varphi}(a, b, c)$. We therefore have that if $\tilde{\varphi} \notin$FO-2S then $\varphi \notin$FO-1BR, showing as claimed that $\varphi \in$FO-1BR if and only if $\tilde{\varphi} \in$FO-2S.

Let us consider Monadic Second Order (MSO) logic; i.e. the extension of first-order logic allowing quantification on subsets. The reader is referred to [24] for a detailed definition of this logic.

The set of modules of a 2-structure is definable by an MSO-formula $MODULES(X)$ in one monadic variable X. Indeed the definition 2 states that X is a module exactly when

$$\forall x \forall y \forall z (x \neq y \wedge x \neq z \wedge x \notin X \wedge y \in X \wedge z \in X \rightarrow (x,y) \equiv (x,z) \wedge (y,x) \equiv (z,x))$$

Let MSO-2SF denotes the MSO-theory of the structures $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$ where $\mathcal{F}$ is a WPSF on S. Our objective is to show the following facts.

(1) The family of modules of $\langle S, \rightleftharpoons_{\mathcal{F}} \rangle$ is $\mathcal{F}$.
(2) The relation $\rightleftharpoons_{\mathcal{F}}$ is MSO-definable in any 2-structure whose set of modules is $\mathcal{F}$.
(3) The class of models $\mathcal{M}$(MSO-2SF) of MSO-2SF is MSO-definable.
(4) MSO-2SF is decidable by interpretation in WMS2S.

The first fact was already shown in Theorem 1. To show the second fact we prove the following easy consequence of the results of section 4.

Theorem 2. *Let $\langle B; \equiv \rangle$ be a 2-structure and $\mathcal{F}$ its set of modules. The relation $\rightleftharpoons_{\mathcal{F}}$ is MSO-definable in $\langle B; \equiv \rangle$.*

proof. By the definition of zigzag and by Lemma 2 we have that $(r,s) \rightleftharpoons_{\mathcal{F}} (u,v)$ if and only if there exists a length 4 $\mathcal{F}$-zigzag between (r,s) and (u,v). We have the result since a length 4 zigzag is clearly definable in MSO. □

The *second-order language* of $\{\equiv\}$ is the extension of its first-order language allowing quantification not just on subsets of the domain but also on relations of any arity. Using quantification on binary relations one can define $\rightleftharpoons_{\mathcal{F}}$ from $\equiv$, since it is the intersection of all equivalence relations which realize $\mathcal{F}$. It is interesting to note that the previous theorem shows that in fact $\rightleftharpoons_{\mathcal{F}}$ can be MSO-defined, so that actually only monadic quantification is necessary.

The third fact, stating that $\mathcal{M}$(MSO-2SF) is MSO-definable, follows directly from the MSO-definability of $\rightleftharpoons_{\mathcal{F}}$. Indeed, let $\Phi(x,y,u,v)$ be the MSO-formula defining $(x,y) \rightleftharpoons_{\mathcal{F}} (u,v)$ in the MSO-language of $\{\equiv\}$. The class of the finite structures $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$ is the class of finite structures $\langle B; \equiv \rangle$ satisfying the axioms of 2-structures with the additional MSO-sentence $\forall x, y \; \Phi(x,y,u,v) \leftrightarrow (x,y) \equiv (u,v)$, stating that $\equiv$ and $\rightleftharpoons_{\mathcal{F}}$ are equal.

In order to prove the last fact, i.e. that MSO-2SF is decidable by interpretation in WMS2S, we will first show in section 6 that MSO-2SF is bi-interpretable with a theory FO-BAF of finite boolean algebra with a distinguished subset, then recall in section 7 some useful facts about the modular decomposition of WPSFs and finally show in section 8 that FO-BAF can be interpreted in WMS2S.

6. Finite boolean algebras

Our objective in this section is to show that MSO-2SF is bi-interpretable with the first-order theory of finite boolean algebras with a distinguished predicate for a WPSF. Let us therefore consider FO-BAF to be the first-order theory of finite boolean algebras in the language $\{\cap, \cup, ^c, 0, 1, F\}$, where $\cap, \cup, ^c, 0, 1$ are the usual boolean algebra operations (X^c is the complement of X) and F is a unary predicate which defines a set satisfying definition 3, where obviously S is replaced by 1, $\emptyset$ by

0 and singletons by the *atoms* (minimal non-zero elements). We will consider that the reader is somewhat familiar with finite boolean algebras. Let us just remind that any finite boolean algebra is isomorphic to the power set of some finite set, that $x \cap y = x$ defines an order $x \leq y$ in a boolean algebra and 0 as well as the *atoms* are definable from this order. Therefore this theory is axiomatized by the usual boolean algebra axioms with the statements of definition 3.

We will first show that MSO-2SF is bi-interpretable with the theory MSO-F of the finite structures $\langle S, F \rangle$, where F is a predicate in one monadic variable defining a WPSF. More precisely we show that for any MSO-sentence φ in the language $\{\equiv\}$ there exists an MSO-sentence φ_F in the language $\{F\}$ such that $\varphi \in$MSO-2SF if and only if $\varphi_F \in$MSO-F and conversely for any MSO-sentence ψ in the language $\{F\}$ there exists an MSO-sentence $\psi_\equiv$ in the language $\{\equiv\}$ such that $\psi \in$ MSO-F if and only if $\psi_\equiv \in$MSO-2SF.

Indeed, one obtains φ_F from φ by replacing $\equiv$ by the definition Ψ of $\rightleftharpoons_{\mathcal{F}}$ in terms of length 4 zigzags, since this notion is based on the WPSF F. Conversely $\psi_\equiv$ is obtained from ψ by replacing F by the MSO-definition of modules $MODULES(X)$.

Now note that for every model $\langle S, F \rangle$ of MSO-F, the structure $\langle \mathcal{P}(S), F \rangle$, where $\mathcal{P}(S)$ is the power set of S, is a model FO-BAF. Conversely for every model $\langle B, F \rangle$ of FO-BAF, the set B is a finite boolean algebra, hence it is isomorphic to $\mathcal{P}(S)$ for some finite set S. We hence have that $\langle S, F \rangle$ is a model of MSO-F, since F is now a monadic predicate.

To show that MSO-2SF is bi-interpretable with FO-BAF it remains to be shown that MSO-F is bi-interpretable with FO-BAF. Take φ an MSO-sentence in the language $\{F\}$. A first-order quantifier can be considered as a quantifier on singletons. An equivalent sentence φ_{BAF} is hence obtained from φ by replacing first-order quantifiers by quantifiers on atoms and monadic quantifiers by first-order ones. Conversely an equivalent sentence φ_F in the MSO language of $\{F\}$ is obtained from a sentence φ of the FO language $\{\cap, \cup, ^c, 0, 1, F\}$ by replacing every quantifier by a monadic quantifier and replacing the operators $\cap, \cup, ^c, 0, 1$ by their MSO-definitions. This transformation is a bi-interpretation by the previous paragraph.

Before we can show how to interpret FO-BAF in WMS2S, we have to recall some facts about the modular decomposition.

7. Modular decomposition of WPSFs

We recall in this section some facts about the modular decomposition of WPSFs which are needed in order to show how to interpret FO-BAF in WMS2S. The reader is referred to Chapter 5 of the book [1] for proofs.

An non-empty element P of a WPSF is said to be *strong* if no X in the WPSF overlaps with P. This means that if some X in the WPSF intersects P then either X is contained in P of X contains P. For P a strong element of a WPSF the maximal strong elements (by inclusion) strictly contained in P form a partition of P, which gives rise to the tree called the *modular decomposition tree* of a WPSF. The nodes of this tree are the strong elements of the WPSF, a strong element P' being a descendant of a strong element P if $P' \subseteq P$. The leaves of this tree are the singletons. Furthermore the strong elements of a WPSF are partitioned in three classes called *trivial*, *linear* and *complete* strong elements respectively. Finally the children of a linear strong element can be ordered in such a way as to make the following characterization of the elements of the WPFS hold.

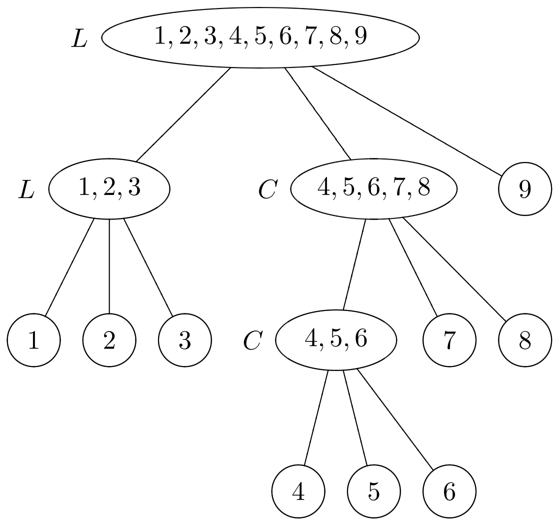

FIGURE 3. A modular decomposition tree

Let S be a set and $\mathcal{F}$ a WPSF on S. For X a non-empty subset of S, there is a smallest (by inclusion) strong element of $\mathcal{F}$ containing X denoted by P_X. A non-empty subset X of S is in $\mathcal{F}$ if and only if all of the following conditions hold.

(1) X is the union of children of P_X.
(2) if P_X is trivial then $X = P_X$.
(3) if P_X is linear then X is the union of an *interval* of children of P_X, which means that $X = P_1 \cup \cdots \cup P_n$ where the P_i's are consecutive children of P_X in the given order (there is no gap).

Note that if P_X is complete, we have that any union of children of P_X is in the WPSF.

Example 2. *Consider the modular decomposition tree of figure 3, where L means that the node is linear, while C means complete. This tree having no trivial node. The modules can easily be enumerated. If* $P_X = \{1,2,3,4,5,6,7,8,9\}$*, then* X *is either* P_X*,* $\{1,2,3,4,5,6,7,8\}$ *or* $\{4,5,6,7,8,9\}$*. If* $P_X = \{1,2,3\}$*, then* X *is either* P_X*,* $\{1,2\}$ *or* $\{2,3\}$*. If* $P_X = \{4,5,6,7,8\}$*, then* X *is either* P_X*,* $\{4,5,6,7\}$*,* $\{4,5,6,8\}$ *or* $\{7,8\}$*. Finally if* $P_X = \{4,5,6\}$ *, then* X *is either* P_X*,* $\{4,5\}$*,* $\{4,6\}$ *or* $\{5,6\}$*.*

The previous conditions give in fact a tree-theoretic characterization of the elements of a WPSF in terms of its modular decomposition. Indeed, since the leaves of the modular decomposition are the singletons, a subset X of S can be considered as a set of leaves. In this way the set P_X contains the leaves which are below the node P_X of the tree. For X a set of leaves, P_X is just the lowest node which is above all elements of X.

We can hence give another characterization of WPSFs this time in terms of trees. For $\mathcal{T}$ a finite tree and n a node of $\mathcal{T}$ we will denote by $[n]$ the set of leaves which are below n. We have the following characterization.

Theorem 3 ([27], see also [18]). *A family of sets containing the empty set is a WPSF if and only if it is a family $\mathcal{F}$ of sets of leaves of a finite, {trivial,linear,complete}-labeled, ordered tree defined in the following way. A non-empty set is in $\mathcal{F}$ if and only if it fulfills all the following conditions.*

1. *X is the union of $[P]$'s for some children P of P_X.*
2. *if P_X is trivial then $X = [P_X]$.*
3. *if P_X is linear then X is the union of an* interval *of children of P_X, which means that $X = [P_1] \cup \cdots \cup [P_n]$ where the P_i's are consecutive children of P_X in the given order (no gap).*

proof. Since [27] is a thesis and [18] covers the case of partitive set families (PSF) we will give here a complete proof.

The left to right direction follows from the previous description of the modular decomposition.

For the converse we have to show that such a family $\mathcal{F}$ satisfies definition 3. First note that from the conditions we have that $[P]$ is always in $\mathcal{F}$ for any node P. We therefore have that S is in $\mathcal{F}$ since it is the set $[P]$ for the root P of the tree. The empty set is there by hypothesis.

It remains therefore to be shown that for overlapping sets X, Y, the sets $X \cup Y$, $X \setminus Y$ and $Y \setminus X$ are in $\mathcal{F}$. Let us first show that if two sets X and Y of leaves overlap then P_X and P_Y are the same node.

Take two sets X and Y of leaves which overlap. We have that P_X and P_Y must be comparable. Without loss of generality let P_Y be a descendant of P_X. X is the union $[P_1] \cup \cdots \cup [P_n]$ of children of P_X. If P_Y is not equal to P_X then P_Y is contained in a unique child P of P_X. Since X and Y intersect, we have that P is among $P_1, \ldots, P_n$ which implies that $Y \subseteq X$, a contradiction with the fact that they must overlap.

It remains to be shown that conditions *(2), (3)* and *(4)* of definition 3 hold. Now if $P_X = P_Y$ is trivial, $X = Y = P_X = P_Y$ so X, Y do not overlap. If $P_X = P_Y$ is complete, we obviously have that $X \cup Y$, $X \setminus Y$ and $Y \setminus X$ are in $\mathcal{F}$. Finally if $P_X = P_Y$ is linear $X \cup Y$, $X \setminus Y$ and $Y \setminus X$ are intervals of children of $P_X = P_Y$, hence again elements of $\mathcal{F}$. □

8. Interpretation of FO-BAF in WMS2S

Consider the infinite structure $\langle \{0,1\}^*; s_0, s_1, \leq, \preccurlyeq \rangle$, where $\{0,1\}^*$ is the set of finite words on 0 and 1, $s_0(x) = x0$ and $s_1(x) = x1$ are the *successors* functions, $x \leq y$ if x is a *prefix* of y, i.e. $xz = y$ for some $z \in \{0,1\}^*$ and finally $x \preccurlyeq y$ if x is *lexicographically* before y, i.e. $x = u0x'$, $y = u1y'$ for some $u, x', y' \in \{0,1\}^*$. The *Weak Monadic Second Order Theory of 2 successors* (WMS2S) is the weak-monadic theory of $\langle \{0,1\}^*; s_0, s_1, \leq, \preccurlyeq \rangle$, i.e. the set of monadic second order sentences satisfied by this structure when monadic quantifiers are interpreted as quantification over *finite* sets. WMS2S has been shown to be decidable by automata on trees by [22, 23].

In this section we will complete our proof that MSO-2SF is decidable by interpretation in the Weak Monadic Second Order Theory of 2 successors (WMS2S), by showing that FO-BAF can be interpreted in WMS2S, by which we mean that for any FO-sentence in the language $\{\cap, \cup,^c, 0, 1, F\}$ there exists an MSO-sentence φ_{WMS2S} such that $\varphi \in$FO-BAF if and only if $\varphi_{WMS2S} \in$WMS2S. The idea of the proof is similar to the method used in [28, Theorem 6.2] to give an interpretation of the theory of finite boolean algebra with a distinguish subalgebra in WMS1S.

Theorem 4. *FO-BAF can be interpreted in WMS2S*

proof. The embedding of FO-BAF in WMS2S interprets a sentence φ of the language $\{\cap, \cup, ^c, 0, 1, F\}$ into the sentence $\forall X \forall X_T \forall X_L \forall X_C\ (\Psi \rightarrow \hat{\varphi})$ of WMS2S, where:

(1) Ψ states that X is a representation of the modular decomposition tree, i.e. a set of binary words representing the nodes of the tree; $y \in X$ being a descendant of $x \in X$ if $x \leq y$.
(2) X_T, X_L, X_C is a partition of X. These sets represent trivial, linear and complete nodes.

This allows, using Theorem 3, to define a predicate F on sets of leaves of X, defining the WPSF arising from this tree. The formula $\hat{\varphi}$ can now be obtained from φ by replacing the quantifiers by monadic quantifiers on sets of leaves of X (this is MSO-definable) and $\cap, \cup, ^c, 0, 1$ by their usual MSO-definitions. Theorem 3 shows that this is indeed an interpretation, completing our proof that MSO-2SF can be interpreted in WMS2S. □

9. Labeled 2-structures

We now turn our attention toward an explicit description of $\rightleftharpoons_{\mathcal{F}}$ and computing its number of equivalence classes. But before, we must recall some important facts on how one can use the decomposition tree in order to construct a 2-structure.

Let $\mathcal{B} = \langle B; \equiv \rangle$ be a 2-structure and $\mathcal{A}$ its modular decomposition tree, which is the modular decomposition tree of its family of modules $\mathcal{F}$. As we said in section 7, the nodes of $\mathcal{A}$ are the strong modules, Q being a descendant of P if $Q \subseteq P$. Furthermore each node of $\mathcal{A}$ is either *trivial, linear* or *complete.*

But what can be said about $\mathcal{B}$'s equivalence relation from the structure of $\mathcal{A}$? As we said after proposition 2 (see also example 3 below), $\mathcal{A}$ does not uniquely determine $\mathcal{B}$. Nevertheless it is well-known (see [1, Chapter 5] and [29] for more details), that at least some information on $\mathcal{B}$ can be determined from $\mathcal{A}$.

Our objective in this section is to recall these facts in order to, in the next section, give a construction from a WPSF $\mathcal{A}$ of a 2-structure realizing $\mathcal{A}$, whose equivalence relation has 6 classes. Furthermore we will also show that a slight modification to this method gives a construction of $\rightleftharpoons_{\mathcal{F}}$, giving also a recurrence to compute its number of classes.

To start with, note that $\mathcal{B}$ induces 2-structures on the nodes of $\mathcal{A}$ in the following way. For $\mathcal{B} = \langle B; \equiv \rangle$ a 2-structure and $X \subseteq B$, let the 2-structure *induced* on X by $\mathcal{B}$ be the 2-structure $\mathcal{B}[X] = \langle X, \equiv_X \rangle$, where $\equiv_X$ is $\equiv$ restricted to X.

Now induced 2-structures pass to the quotient in the following way. For P a node of $\mathcal{A}$ (i.e. a strong element of $\mathcal{F}$) denote by $\mathcal{P}_P$ the set of maximal strong modules included in P (these are the children of P in $\mathcal{A}$). This set forms a partition of P. Denote by $\mathcal{B}[P]/\mathcal{P}_P$ the *quotient* of $\mathcal{B}[P]$ by $\mathcal{P}_P$. This is the 2-structure whose underlying set is $\mathcal{P}_P$ and whose equivalence relation fulfills $(P_1, P_2) \equiv (P_3, P_4)$ if $(p_1, p_2) \equiv (p_3, p_4)$ in $\mathcal{B}$ for some $p_1 \in P_1, p_2 \in P_2, p_3 \in P_3, p_4 \in P_4$. This is well-defined since the elements of $\mathcal{P}_P$ are modules.

In fact, if P is either linear or complete $\mathcal{B}[P]/\mathcal{P}_P$ can be simply described. Before we describe this structure, recall that one of our objectives is to construct

2-structures from a decomposition tree. A natural idea would be to endorse each $\mathcal{P}_P$, for P a node of $\mathcal{A}$ with a 2-structure, but this will not work for the following reason.

The quotients $\mathcal{B}[P]/\mathcal{P}_P$, for P a node of $\mathcal{A}$, are not enough to recover the 2-structure $\mathcal{B}$. For instance consider a modular decomposition tree with root P having children $P_1, \ldots, P_n$. The 2-structure $\mathcal{B}[P]/\mathcal{P}_P$ determine the equivalence relation on distinct pair spanning two different children P_i, P_j $(i \neq j)$, while the 2-structures $\mathcal{B}[P_1]/\mathcal{P}_{P_1}, \ldots, \mathcal{B}[P_n]/\mathcal{P}_{P_n}$ determine the equivalence for pairs which are within the same P_i. Now two children P_i, P_j $(i \neq j)$, could each have a single equivalence class for its distinct pairs and we have no way from these structures to know if these are distinct classes in $\mathcal{B}$ or not.

The way to overcome this problem is to label each equivalence class. This will allow comparing classes of different quotients.

Another benefit of these labels is to simplify the description of $\mathcal{B}[P]/\mathcal{P}_P$ for P linear or complete. In fact [30] (see also [1, Theorem 8.5], [31] and [29, Theorem 3]) shows that linear and complete nodes can be simply characterized in terms of the 2-structure's equivalence. Let us first introduce some definitions.

Definition 6. *A* labeled 2-structure of rank k *(also called a* binary structure *in [29]) is a structure $\mathcal{L} = \langle L; r \rangle$ where L is a finite set and r (the* rank-function*) is a function from the set of distinct pairs of elements of L into $\{0, \ldots, k-1\}$.*

A labeled 2-structure $\mathcal{L} = \langle L; r \rangle$ determines a unique 2-structure $\langle L; \equiv \rangle$ (which we will call the *underlying* 2-structure), where $(a, b) \equiv (c, d)$ if and only if $r(a, b) = r(c, d)$. Conversely one can associate to any 2-structure $\mathcal{B} = \langle B; \equiv \rangle$ a labeled 2-structure by giving a different rank to every equivalence class of $\equiv$. In this case this association is not unique since any ranking of the classes will do, as long as the rank is as great as the number of classes of $\equiv$ and different equivalence classes are associate to different values.

A module of a labeled 2-structure is a module of its underlying 2-structure or equivalently, for $\mathcal{L} = \langle L; r \rangle$ a labeled 2-structure, a *module* of $\mathcal{L}$ is a subset M of L such that, for all $x \notin M$, $y, z \in M$, $r(x, y) = r(x, z)$ and $r(y, x) = r(z, x)$.

The family of modules of a labeled 2-structure forms a WPSF and, as before, we can speak of the modular decomposition tree $\mathcal{A}$ of a labeled 2-structure $\mathcal{L}$. Furthermore exactly as for 2-structures, the labeled 2-structure $\mathcal{L}$ induces a labeled 2-structure $\mathcal{L}[P]/\mathcal{P}_P$ for P a strong module, by defining $r(P_1, P_2) = r(p_1, p_2)$ for some $p_1 \in P_1, p_2 \in P_2$. This is again well-defined since the elements of $\mathcal{P}_P$ are modules.

We now come to the linear and complete labeled 2-structures characterizing linear and complete nodes of the decomposition tree.

Definition 7. *A labeled 2-structure $\mathcal{L} = \langle L; r \rangle$ of rank k is said to be (a, b)-*linear *if a, b are distinct elements of $\{0, \ldots, k-1\}$ such that $\{(x, y) \in B; r(x, y) = a$ and $r(y, x) = b\}$ defines a total linear order on L.*

Definition 8. *A labeled 2-structure $\mathcal{L} = \langle L; r \rangle$ of rank k is said to be a-*complete *if a is an element of $\{0, \ldots, k-1\}$ such that $\{(x, y) \in L; r(x, y) = a$ and $r(y, x) = a\}$ defines a complete graph (it is equal to the set of distinct pairs of L) on L.*

We now have the following characterization.

Theorem 5 ([30], see also [1, Theorem 8.5], [31] and [29, Theorem 3])**.** *Let* $\mathcal{L} = \langle L, r\rangle$ *be a labeled 2-structure of rank* k *and* P *a strong module of* $\mathcal{L}$ *such that* $|\mathcal{L}[P]/\mathcal{P}_P| \geq 3$. *We then have that:*

(1) P *is linear if and only if* $\mathcal{L}[P]/\mathcal{P}_P$ *is* (a,b)*-linear for some* $a, b \in \{0, \ldots, k-1\}$.
(2) P *is complete if and only if* $\mathcal{L}[P]/\mathcal{P}_P$ *is* a*-complete for some* $a \in \{0, \ldots, k-1\}$.

As we already noted just before Proposition 3, if $\mathcal{L}$ is a labeled 2-structure on a two element set $L = \{l_1, l_2\}$ there is only one possibility for its set of modules since there is only one WPSF on L. But if $r(l_1, l_2) \neq r(l_2, l_1)$, then $\mathcal{L}$ is (a,b)-ordered. Otherwise $r(l_1, l_2) = r(l_2, l_1)$ and $\mathcal{L}$ is a-complete. In the construction of $\mathcal{L}_{3,\mathcal{F}}$ below, two element sets can be treated either way. In the construction of $\mathcal{L}_{\mathcal{F}}$ they will be treated as (a,b)-ordered in order to have the maximal number of classes.

The previous theorem leaves the remaining case P trivial open. In fact in that last case, there are many possibilities for the structure of $\mathcal{L}[P]/\mathcal{P}_P$. Let us give a simple example of two labeled 2-structures $\mathcal{L}$ and $\mathcal{L}'$ on the same set $\{1, \ldots, n\}$, both having no non-trivial modules. Both structures will have the same decomposition tree, which is formed of two levels, one containing the root, and the second the singletons. It hence will follow that $\mathcal{L}[L]/\mathcal{P}_L = \mathcal{L}$ and $\mathcal{L}'[L']/\mathcal{P}_{L'} = \mathcal{L}'$, so the roots are trivial nodes.

Example 3. *Take* $\mathcal{L}$ *to be an* ordered-path-graph, *i.e.* $\mathcal{L} = \langle\{1, \ldots, n\}; r\rangle$, $r(i, i+1) = 1$ *for* $i = 1, \ldots, n-1$ *and* $r(x,y) = 0$ *otherwise. One easily checks that* $\mathcal{L}$ *has no non-trivial modules.*

For the second structure take the trivial *labeled 2-structure, which is* $\mathcal{L}' = \langle\{1, \ldots, n\}; r'\rangle$, *with* r' *giving a different value for different distinct pairs. This structure has rank* $n(n-1)$. *Here again one easily checks that* $\mathcal{L}'$ *has no non-trivial module.*

The labeled 2-structures $\mathcal{L}$ and $\mathcal{L}'$ hence both realize the trivial WPSF formed of the trivial modules. Furthermore these structures have non-isomorphic underlying 2-structures when $n \geq 3$.

To summarize, from a labeled 2-structure $\mathcal{L}$ one gets induced labeled 2-structures $\mathcal{L}_P = \mathcal{L}[P]/\mathcal{P}_P$ on $\mathcal{P}_P$ for all nodes P of its decomposition tree $\mathcal{A}$. Now from the decomposition tree and the $\mathcal{L}_P$'s one can reconstruct easily $\mathcal{L}$ in the following way. Take $x, y \in L$, and let $P_{x,y}$ be the *least upper bound* of x, y, which is the smallest (by inclusion) strong module which contains both x and y. Now x, y are contained in two different children X, Y of $P_{x,y}$ (since otherwise $X = Y$ would contradict the minimality of $P_{x,y}$). Now $r(x,y)$ is equal to the value of $r(X,Y)$ in $\mathcal{L}_{P_{x,y}}$. Furthermore, if P is linear or complete the structure of $\mathcal{L}_P$ is known.

In the next section we will show, conversely, how one can start from a decomposition tree $\mathcal{A}$, add labeled 2-structures on the $\mathcal{P}_P$'s, in order to construct labeled 2-structures having $\mathcal{A}$ as their modular decomposition tree.

10. Realizing WPSF with labeled 2-structures

Our objective in this section is to give a construction from a WPSF $\mathcal{F}$ of a labeled 2-structure realizing $\mathcal{F}$, whose rank is 3 and whose underlying equivalence relation has 6 classes. This fact is well known and appears in [1, Exercise 5.5]. We thank Pierre Ille for making us aware of this construction. We will conclude the section by showing that one cannot, in general, realizes a WPSF by a 2-structure having less than 6 classes, hence that there are WPSF which are not the family of modules of some graph.

Instead of simply giving a construction of a rank 3 labeled 2-structure from the modular decomposition tree of a WPSF, we will give a slightly more general construction which also allows the construction of $\rightleftharpoons_{\mathcal{F}}$ from a WPSF $\mathcal{F}$. This will also give a recurrence to compute the number of classes of $\rightleftharpoons_{\mathcal{F}}$ from the decomposition tree.

In order to give our construction we will need the following labeled 2-structures.

Definition 9. *A labeled* 2*-structure* $\mathcal{L}$ *is said to be an* (a, b)-ordered-path-graph *if* a, b *are distinct elements of* $\{0, \ldots, k-1\}$ *and* $\mathcal{L} = \langle\{1, \ldots, n\}; r\rangle$*, where* $r(i, i+1) = a$ *for* $i = 1, \ldots, n-1$ *and* $r(x, y) = b$ *otherwise.*

Again one easily checks that an (a, b)-ordered-path-graph has no non-trivial modules.

From a WPSF $\mathcal{F}$ on the set S, we will define from its modular decomposition tree $\mathcal{A}$ two labeled 2-structures $\mathcal{L}_{3,\mathcal{F}}$ and $\mathcal{L}_{\mathcal{F}}$. The labeled 2-structure $\mathcal{L}_{3,\mathcal{F}}$ will be of rank 3 while the rank of $\mathcal{L}_{\mathcal{F}}$ will be recursively computable from $\mathcal{A}$. Furthermore we will show the following things.

1. The number of equivalence classes of $\mathcal{L}_{3,\mathcal{F}}$'s underlying 2-structure is 6.
2. The number of equivalence classes of $\mathcal{L}_{\mathcal{F}}$'s underlying 2-structure is maximal among all 2-structures realizing $\mathcal{F}$.
3. $\mathcal{L}_{\mathcal{F}}$'s underlying 2-structure is $\langle S; \rightleftharpoons_{\mathcal{F}}\rangle$.
4. The family of modules of both $\mathcal{L}_{3,\mathcal{F}}$ and $\mathcal{L}_{\mathcal{F}}$ is $\mathcal{F}$.

The general method is, in order to define a labeled 2-structure, to proceed as follows. Starting from the modular decomposition tree $\mathcal{A}$ of some WPSF $\mathcal{F}$ on a set S, define for every node P of $\mathcal{A}$ a labeled 2-structure $\langle \mathcal{P}_P; r_P\rangle$ on $\mathcal{P}_P$. Now define a labeled 2-structure $\langle S; r\rangle$ on S by letting $r(x, y) = r_{P_{x,y}}(X, Y)$, where, as before, $P_{x,y}$ is the smallest (by inclusion) strong element which contains both x and y while X, Y are the children of $P_{x,y}$ containing x, y respectively. We will show that under suitable hypothesis, $\mathcal{F}$ is the family of modules of $\langle S; r\rangle$.

Let us first define our labeled 2-structure $\mathcal{L}_{3,\mathcal{F}}$.

Our construction moves top-down from the root to the leaves. For Q a node of the modular decomposition tree $\mathcal{A}$, we define a labeled 2-structure $\mathcal{L}_{3,Q,\mathcal{F}}$ on $\mathcal{P}_Q$. In order to define $\mathcal{L}_{3,Q,\mathcal{F}}$, we consider that $\mathcal{L}_{3,P,\mathcal{F}}$ has been defined for P the parent of Q. Furthermore we make sure to have the following properties.

1. if Q is trivial then $\mathcal{L}_{3,Q,\mathcal{F}}$ will be either a $(0, 1)$- or a $(1, 2)$-ordered-path-graph.
2. if Q is linear then $\mathcal{L}_{3,Q,\mathcal{F}}$ will be either $(0, 1)$- or $(1, 2)$-linear.
3. if Q is complete then $\mathcal{L}_{3,Q,\mathcal{F}}$ will be either 1- or 2-complete.

Say that a distinct pair (x, y) is a (a, b)-*pair* in some labeled 2-structure $\langle S; r\rangle$, if $r(x, y) = a$ and $r(y, x) = b$. Note that here a and b can be equal. Let Q be a node of $\mathcal{A}$ and P its parent. We define $\mathcal{L}_{3,Q,\mathcal{F}}$ as follows.

1. If Q is trivial, let $\mathcal{L}_{3,Q,\mathcal{F}}$ be a $(0, 1)$-ordered-path-graph if P contains no $(0, 1)$- or $(1, 1)$-pair and let it be a $(1, 2)$-ordered-path-graph otherwise.
2. If Q is linear, let $\mathcal{L}_{3,Q,\mathcal{F}}$ be a $(0, 1)$-linear labeled 2-structure if P contains no $(0, 1)$-pair and let it be a $(1, 2)$-linear labeled 2-structure otherwise. In both case the ordering must be as for $Q_{\mathcal{P}}$ in $\mathcal{A}$.

(3) If Q is complete, let $\mathcal{L}_{3,Q,\mathcal{F}}$ be a 1-complete labeled 2-structure if P contains no $(1,1)$-pair and let it be a 2-complete labeled 2-structure otherwise.

Note that we only use labels $0,1$ and 2 so that $\mathcal{L}_{3,Q,\mathcal{F}}$ is of rank 3. Note also that a $(0,1)$-ordered-path-graph contains only $(0,1)$- $(1,0)$- and $(1,1)$-pairs; a $(1,2)$-ordered-path-graphs contains only $(1,2)$- $(2,1)$- and $(2,2)$-pairs; a $(0,1)$-linear labeled 2-structures contains only $(0,1)$ or $(1,0)$-pairs; a $(1,2)$-linear labeled 2-structures contains only $(1,2)$ or $(2,1)$-pairs; a 1-complete labeled 2-structures contains only $(1,1)$-pairs; while a 2-complete labeled 2-structure contains only $(2,2)$-pairs. There are therefore 6 equivalence classes in the underlying 2-structure, which correspond to $(r(x,y),r(y,x))$ equal to $(0,1),(1,0),(1,1),(1,2),(2,1),(2,2)$ respectively. Furthermore we never have a parent P and a child Q such that both $\mathcal{L}_{3,Q,\mathcal{F}}$ and $\mathcal{L}_{3,P,\mathcal{F}}$ contain an (a,b)-pair for the same a and b.

Now for $\mathcal{L}_{\mathcal{F}}$. This time we work recursively on the structure of $\mathcal{A}$, i.e. from the leaves to the root. We construct $\mathcal{L}_{P,\mathcal{F}} = \langle \mathcal{P}_P; r_{P,\mathcal{F}}\rangle$ considering that the labeled 2-structures $\mathcal{L}_{P_1,\mathcal{F}},\ldots,\mathcal{L}_{P_n,\mathcal{F}}$ have already been defined for $P_1,\ldots,P_n$ the children of P. We also consider that no two of the subtrees rooted at $P_1,\ldots,P_n$ share common labels. This can be achieved easily; it suffices that the recursions on the different P_i's uses different pools of labels.

Now define $\mathcal{L}_{P,\mathcal{F}}$ in the following way.

(1) If P is trivial, let $\mathcal{L}_{P,\mathcal{F}}$ be a trivial labeled 2-structure, using new labels which appear in none of $P_1,\ldots,P_n$.
(2) If P is linear, let $\mathcal{L}_{P,\mathcal{F}}$ be an (a,b)-linear labeled 2-structure ordering $P_{\mathcal{P}}$ as it is in $\mathcal{A}$, where a,b are two new labels which appear in none of $P_1,\ldots,P_n$.
(3) If P is complete, let $\mathcal{L}_{P,\mathcal{F}}$ be an a-complete labeled 2-structure, where a is a new label which appears in none of $P_1,\ldots,P_n$.

Here also we never have a parent P and a child Q such that both $\mathcal{L}_{Q,\mathcal{F}}$ and $\mathcal{L}_{P,\mathcal{F}}$ contain an (a,b)-pair for the same a and b.

The number of labels used in $\mathcal{L}_{\mathcal{F}}$ can easily be computed from $\mathcal{A}$ using the following recurrence, where P has n children $P_1,\ldots,P_n$.

$$no\_labels(\mathcal{L}_{\mathcal{F}}[P]) = \begin{cases} n(n-1) & +\sum_{i=1}^{n} no\_labels(\mathcal{L}_{\mathcal{F}}[P_i]) \quad \text{for } P \text{ trivial} \\ 2 & +\sum_{i=1}^{n} no\_labels(\mathcal{L}_{\mathcal{F}}[P_i]) \quad \text{for } P \text{ linear} \\ 1 & +\sum_{i=1}^{n} no\_labels(\mathcal{L}_{\mathcal{F}}[P_i]) \quad \text{for } P \text{ complete} \end{cases}$$

The total number of labels of $\mathcal{L}_{\mathcal{F}}$ being $no\_labels(\mathcal{L}_{\mathcal{F}}[R])$ for R the root of $\mathcal{A}$. Note that the number of classes of the underlying 2-structure is equal to this same value, since in the first case we add $n(n-1)$, in the second case 2 and in the last case 1 new equivalence classes.

Furthermore we have the following result.

Proposition 4. *Let $\mathcal{A}$ be a modular decomposition tree of a WPSF $\mathcal{F}$. For P a node of $\mathcal{A}$, let $P_1,\ldots,P_n$ be the children of P in $\mathcal{A}$. Let $\mathcal{B}$ be a 2-structure realizing $\mathcal{F}$. We then have that*

$$no\_classes(\mathcal{B}[P]) \leq \begin{cases} n(n-1) & +\sum_{i=1}^{n} no\_classes(\mathcal{B}[P_i]) \quad \textit{for } P \textit{ trivial} \\ 2 & +\sum_{i=1}^{n} no\_classes(\mathcal{B}[P_i]) \quad \textit{for } P \textit{ linear} \\ 1 & +\sum_{i=1}^{n} no\_classes(\mathcal{B}[P_i]) \quad \textit{for } P \textit{ complete} \end{cases}$$

where $no\_classes(\mathcal{B})$ is the number of equivalence classes of the 2-structure $\mathcal{B}$.

proof. Note that here a two element set must be treated as linear. Furthermore in order to have (see corollary 2 below) that $\mathcal{L}_{\mathcal{F}}$ has the maximal number of classes, two elements sets must also be treated as linear in the definition of $\mathcal{L}_{\mathcal{F}}$.

Let P be a node of $\mathcal{A}$. The number of equivalence classes for distinct pairs in $\mathcal{B}[P]$ is bounded by $no\_classes(\mathcal{B}[P]/\mathcal{P}_P) + \sum_{i=1}^{n} no\_classes(\mathcal{B}[P_i])$. If P has n children, then $no\_classes(\mathcal{B}[P]/\mathcal{P}_P) \leq n(n-1)$ so the first case holds. If P is linear then $\mathcal{B}[P]/\mathcal{P}_P$ is (a,b)-ordered by Theorem 5, hence $no\_classes(\mathcal{B}[P]/\mathcal{P}_P) \leq 2$ and the second case holds. For the third case, note that if P is complete then again by Theorem 5, $\mathcal{B}[P]/\mathcal{P}_P$ is a-complete, hence $no\_classes(\mathcal{B}[P]/\mathcal{P}_P) \leq 1$. □

Corollary 2. *Let $\mathcal{A}$ be the decomposition tree of a WPSF $\mathcal{F}$ on a set S. The underlying 2-structure of $\mathcal{L}_{\mathcal{F}}$ is $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$.*

proof. We showed in Theorem 1 that $\rightleftharpoons_{\mathcal{F}}$ is the finer equivalence relation realizing $\mathcal{F}$, it therefore has the maximal number of classes among the equivalence relations realizing $\mathcal{F}$. The statement hence now follows from the previous proposition. □

It is now left to show that the family of modules of both $\mathcal{L}_{3,\mathcal{F}}$ and $\mathcal{L}_{\mathcal{F}}$ is $\mathcal{F}$. The crucial observation is to note that we never have in our constructions nodes P, Q such that Q is a child of P, both of $\langle \mathcal{P}_P; r_P \rangle$, $\langle \mathcal{P}_Q; r_Q \rangle$ containing an (a,b)-pair for some labels a, b.

Theorem 6. *Let $\mathcal{A}$ be the modular decomposition tree of a WPSF $\mathcal{F}$ on the set S. Let $\mathcal{L}_P = \langle \mathcal{P}_P; r_P \rangle$, for P a node of $\mathcal{A}$, be a labeled 2-structure, such that the following conditions hold.*

1. *if P is trivial then $\mathcal{L}_P$ is* indecomposable, *i.e. it has no non-trivial module.*
2. *if P is linear then $\mathcal{L}_P$ is (a,b)-linear for some labels a, b, ordering the children of P as in $\mathcal{A}$.*
3. *if P is complete then $\mathcal{L}_P$ is a-complete for some label a.*

Suppose finally that for all P, Q, Q a child of P, $\mathcal{L}_P$ and $\mathcal{L}_Q$ never both contain an (a,b)-pair for some labels a and b.

Let the labeled 2-structure $\mathcal{L} = \langle S; r \rangle$ on S be defined by letting $r(x,y) = r_{P_{x,y}}(X,Y)$, where, $P_{x,y}$ is the smallest (by inclusion) strong element which contains both x and y while X, Y are the children of $P_{x,y}$ containing x, y respectively. We then have that the family of modules of $\langle S; r \rangle$ is $\mathcal{F}$.

proof. Let M be a subset of S. Let P_M be the smallest strong element of $\mathcal{F}$ (i.e. a node of $\mathcal{A}$) containing M.

We will first show that if $M \in \mathcal{F}$ then M is a module of $\mathcal{L}$. First note that from the definition of $\mathcal{L}$ an element outside of P_M cannot distinguish elements of M, i.e. for all $x \notin P_M$, $y, z \in M$, $r(x,y) = r(x,z)$ and $r(y,x) = r(z,x)$. So in order to show that M is a module, it will be sufficient to show that

Claim 1. *Elements of $P_M \setminus M$ don't distinguish elements of M.*

Now, if P_M is trivial, then $M = P_M$ and the claim is true. If P_M is complete, then M is the union of some P_i's and since $\mathcal{L}_{P_M}$ is a-complete for some a, an element of a P_j not contained in M cannot distinguish between elements of M. Finally if P_M is linear then $\mathcal{L}_{P_M}$ is (a,b)-ordered, ordering $P_{\mathcal{P}}$ as it is in $\mathcal{A}$, and again the claim is satisfied.

Conversely, let M be a module of $\mathcal{L}$ and let us show that $M \in \mathcal{F}$. First, if $M = P_M$ obviously $M \in \mathcal{F}$. Otherwise by definition of P_M, M must intersects at

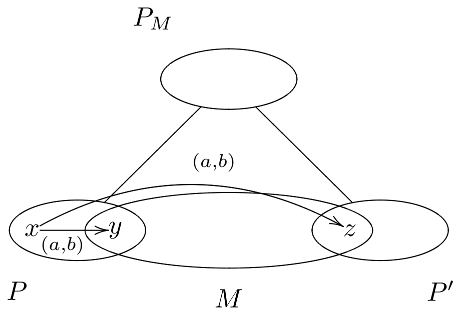

FIGURE 4. Theorem 6's decomposition tree

least two distinct children of P_M. Let us show that a child P of P_M which intersects M must be included in M. Suppose that it is not the case. Take then (see figure 4) $x \in P \setminus M$, $y \in P \cap M$ such that x, y are in two different children of P. This is possible since otherwise $P \setminus M$ and $P \cap M$ would be in a single child of P, hence $P = P \setminus M \cup (P \cap M)$ would be in a single child of P which is impossible. Take also $z \in M \cap P'$ for P' another child of P_M which intersects M.

Now since M is a module of $\mathcal{L}$ it follows that $r(x,y) = r(x,z) = a$ and $r(y,x) = r(z,x) = b$, so both (x,y) and (x,z) are (a,b)-pairs. Since x and y are members of distinct children X, Y of P, by definition of $\mathcal{L}$ we also have that (X,Y) is an (a,b)-pair of $\mathcal{L}[P]/\mathcal{P}_P$. We also have that x, z are respectively elements of the distinct children P and P' of P_M hence (P,P') is an (a,b)-pair of $\mathcal{L}[P_M]/\mathcal{P}_P$, contradicting the hypothesis of the theorem.

We therefore have that $M = \bigcup_{i=1}^{n} P_i$ for $P_1, \ldots, P_n$ some children of P_M. Furthermore since M is a module of $\mathcal{L}$, $\{P_1, \ldots, P_n\}$ is a module of $\mathcal{L}_{P_M}$. Now by hypothesis $\{P_1, \ldots, P_n\}$ contains at least two elements, so $\mathcal{L}_{P_M}$ cannot be trivial. If $\mathcal{L}_{P_M}$ is linear then $P_1, \ldots, P_n$ are consecutive children of P_M and $M \in \mathcal{F}$. Finally if $\mathcal{L}_{P_M}$ is complete, then obviously $M \in \mathcal{F}$. $\square$

Let us conclude this section by showing that one cannot, in general, realizes a WPSF by a 2-structure having less than 6 classes. Consider the WPSF $\mathcal{F}$ of example 2 (Figure 3). Since any 2-structure is the underlying 2-structure of a labeled 2-structure, it is sufficient to show that any labeled 2-structure realizing $\mathcal{F}$ has an underlying 2-structure with at least 6 different equivalence classes. By Theorem 5 we know that in any labeled 2-structure $\mathcal{L}$ realizing $\mathcal{F}$, the linear nodes $\{1,2,3,4,5,6,7,8,9\}$ and $\{1,2,3\}$ (in fact $\mathcal{L}[P]/\mathcal{P}_P$, for $P = \{1,2,3,4,5,6,7,8,9\}$ and $\{1,2,3\}$) must be respectively (a,b)- and (c,d)-linear. Now $(a,b) \neq (c,d)$ since otherwise $\{3,4,5,6,7,8,9\}$ would be in $\mathcal{F}$. Similarly $(a,b) \neq (d,c)$ since otherwise $\{1,4,5,6,7,8,9\}$ would be in $\mathcal{F}$. We therefore have at least 4 equivalent classes corresponding to (a,b)-, (b,a)-, (c,d)- and (d,c)-pairs.

Again by Theorem 5 we know that the complete nodes $\{4,5,6,7,8\}$ and $\{4,5,6\}$ must be respectively e- and f-complete. Here again $e \neq f$ since otherwise $\{4,7\}$ would be in $\mathcal{F}$. We therefore have two more equivalence classes corresponding to (e,e)- and (f,f)-pairs, bringing the total to at least 6 classes.

11. Symmetric 2-structures and Partitive Set Families

A 2-structure $\langle B, \equiv \rangle$ is said to be *symmetric* if for all $x, y \in B$, we have that $(x, y) \equiv (y, x)$. A non-oriented graph gives rise to a symmetric 2-structure, hence this notion is a generalization of non-orientation from graphs to 2-structures.

The definition of module can be simplified for symmetric 2-structures: for a symmetric 2-structure $\langle B, \equiv \rangle$, a subset M of B is a *module* if and only if for all $x \in B \setminus M$ and all $y, z \in M$, it holds that $(x, y) \equiv (x, z)$.

WPSF describe set-theoretically the family of modules of 2-structures. Similarly one can describe the family of modules of symmetric 2-structures in the following way. Let $X \Delta Y$ denote the symmetric difference of the set X and Y, which is $(X \cup Y) \setminus (X \cap Y)$.

Definition 10 (Partitive Set Families)**.** *A* partitive set family *(PSF) on some set S is a family of subsets $\mathcal{F}$ of S such that*

(1) *S, $\emptyset$ and all singletons are in $\mathcal{F}$.*
(2) *$X \cap Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$.*
(3) *$X \cup Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$ and overlap.*
(4) *$X \setminus Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$ and overlap.*
(5) *$X \Delta Y$ is in $\mathcal{F}$, if X, Y are in $\mathcal{F}$ and overlap.*

A set family is therefore a PSF if it is a WPSF and furthermore it fulfills the last condition of the previous definition.

As we said before, [1, Lemma 3.5] shows that any modular set family is a WPSF. Furthermore if $\mathcal{F}$ is the family of modules of a symmetric 2-structure, then $\mathcal{F}$ also satisfies the last condition of the definition and is hence a PSF. This can be seen as follows.

Consider a symmetric 2-structure $\langle B, \equiv \rangle$ whose family of modules is $\mathcal{M}$. We have to show that for any $X, Y \in \mathcal{M}$, $X \Delta Y$ is in $\mathcal{M}$. This means that we have to show that for any $x \notin X \Delta Y$, and any $y, z \in X \Delta Y$, it holds that $(x, y) \equiv (x, z)$.

Take such x, y, z. If $y, z \in X$, then these elements are both in $X \setminus Y$. Since this last set avoids x and is in $\mathcal{M}$ by the fourth condition, it follows that $(x, y) \equiv (x, z)$ and the claim is shown. The case $y, z \in Y$ can be handled in a similar way.

We are then left to consider the case where one of y, z is in X and the other in Y. We can suppose without loss of generality that $y \in X$, $z \in Y$. If $x \notin X \cup Y$ then the claim follows from the fact that $X \cup Y$ is in $\mathcal{M}$. Otherwise $x \in X \cap Y$. In that case we have that $(z, x) \equiv (z, y)$ since $X \in \mathcal{M}$ and $z \notin X$. Similarly we have that $(y, x) \equiv (y, z)$ since $Y \in \mathcal{M}$ and $y \notin Y$. It hence follows that $(x, y) \equiv (y, x) \equiv (y, z) \equiv (z, y) \equiv (z, x) \equiv (x, z)$, completing the argument.

Since $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$ realizes $\mathcal{F}$ for any WPSF $\mathcal{F}$, so in particular for a PSF $\mathcal{F}$, it would be a nice if $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$ was furthermore symmetric when $\mathcal{F}$ is a PSF. Unfortunately this is not the case as shown by the following example.

Example 4. *Let $S = \{a, b, c, d\}$. Consider the set family $\mathcal{F}$ containing in addition to S, the empty set and the singletons, the two sets $\{a, b\}$ and $\{c, d\}$. One easily checks that $\mathcal{F}$ is a PSF. To show that $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$ is not symmetric, it is sufficient to show that $(b, c) \rightleftharpoons_{\mathcal{F}} (c, b)$ does not hold. This is the case since otherwise, by Lemma 2 we would have a length 4 zigzag b, c, x, b, c, for x some element of S. But in that case, by the definition of zigzag, we would have an $X \in \mathcal{F}$ containing b, c but not x, therefore $X \neq S$. But there is no such X in $\mathcal{F}$.*

We will therefore consider the *symmetric closure* defined as follows.

Definition 11 (symmetric closure)**.** *The* symmetric closure *of a relation* $\equiv$ *on distinct pairs of elements of some set* S *is the relation* $\equiv_{sc}$*, where* $(a, b) \equiv_{sc} (c, d)$ *holds if either* $(a, b) \equiv (c, d)$ *or* $(a, b) \equiv (d, c)$*.*

A 2-structure $\langle S, \equiv\rangle$ is said to be *reversible*, if $(x, y) \equiv (u, v)$ if and only if $(y, x) \equiv (v, u)$. Recall that in remark 1 we noted that $(a, b) \rightleftharpoons_{\mathcal{F}} (c, d)$ if and only if $(b, a) \rightleftharpoons_{\mathcal{F}} (d, c)$. Therefore for any WPSF $\mathcal{F}$ on the set S the 2-structure $\langle S, \rightleftharpoons_{\mathcal{F}}\rangle$ is *reversible*. We now have the following simple result.

Proposition 5. *If* $\langle B, \equiv\rangle$ *is a reversible* 2*-structure, then* $\equiv_{sc}$ *is an equivalence relation and hence* $\langle S, \equiv_{sc}\rangle$ *is a* 2*-structure, which we will call the* symmetric closure *of* $\langle S, \equiv\rangle$*.*

proof. We have to show that $\equiv_{sc}$ is reflexive, symmetric and transitive. That this relation is reflexive and symmetric follows directly from the same properties for $\equiv$. It remains to be shown that $\equiv_{sc}$ is transitive.

Let therefore $(a, b) \equiv_{sc} (c, d)$ and $(c, d) \equiv_{sc} (e, f)$. We have to show that $(a, b) \equiv_{sc} (e, f)$. Now $(a, b) \equiv_{sc} (c, d)$ means that $(a, b) \equiv (c, d)$ or $(a, b) \equiv (d, c)$ holds while $(c, d) \equiv_{sc} (e, f)$ means that $(c, d) \equiv (e, f)$ or $(c, d) \equiv (f, e)$ holds.

If $(a, b) \equiv (c, d)$, then by transitivity of $\equiv$ we have that $(a, b) \equiv (e, f)$ or $(a, b) \equiv (f, e)$ and $(a, b) \equiv_{sc} (e, f)$ and the result is shown. If $(a, b) \equiv (d, c)$, then we use the fact that $\equiv$ is reversible, so either $(d, c) \equiv (f, e)$ or $(d, c) \equiv (e, f)$ holds. Again by transitivity of $\equiv$ we get that $(a, b) \equiv (f, e)$ or $(a, b) \equiv (e, f)$ and $(a, b) \equiv_{sc} (e, f)$ completing the proof. □

For $\mathcal{F}$ a PSF, we will denote by $\Leftrightarrow_{\mathcal{F}}$ the symmetric closure of $\rightleftharpoons_{\mathcal{F}}$.

Proposition 6. *Let* $\mathcal{F}$ *be the family of modules of some symmetric* 2*-structure* $\langle B; \equiv\rangle$*. The relation* $\Leftrightarrow_{\mathcal{F}}$ *is finer than* $\equiv$ *(i.e.* $\Leftrightarrow_{\mathcal{F}} \subseteq \equiv$*).*

proof. From Proposition 3 we have that $\rightleftharpoons_{\mathcal{F}}$ is finer than $\equiv$. Now since $\langle B; \equiv\rangle$ is a symmetric 2-structure, it follows that the symmetric closure $\Leftrightarrow_{\mathcal{F}}$ of $\rightleftharpoons_{\mathcal{F}}$ is finer than $\equiv$, completing the proof. □

As for $\rightleftharpoons_{\mathcal{F}}$, by the last proposition, for any symmetric 2-structure $\mathcal{B} = \langle B; \equiv\rangle$ which realizes $\mathcal{F}$ we have that $\Leftrightarrow_{\mathcal{F}}$ is finer that $\equiv$. If we furthermore had that $\Leftrightarrow_{\mathcal{F}}$ realizes $\mathcal{F}$, then $\Leftrightarrow_{\mathcal{F}}$ would be the finer such relation and hence it would be the intersection of all symmetric binary equivalence relations realizing $\mathcal{F}$. We will now show that this is indeed the case.

We will now prove that for a PSF $\mathcal{F}$, the modules of $\langle S; \Leftrightarrow_{\mathcal{F}}\rangle$ are exactly the elements of $\mathcal{F}$.

Theorem 7. *Let* $\mathcal{F}$ *be a PSF on* S*. The family of modules of* $\langle S; \Leftrightarrow_{\mathcal{F}}\rangle$ *is equal to* $\mathcal{F}$*. Furthermore* $\Leftrightarrow_{\mathcal{F}}$ *is the intersection of all symmetric equivalence relations which realize* $\mathcal{F}$*.*

We will then have the following immediate corollary, where a *symmetric modular set family* is the family of modules of some symmetric 2-structure.

Corollary 3. *Any PSF is a symmetric modular set family.*

Before we can prove Theorem 7, we will need the following lemma.

Table 2. Conditions on the Sets of Lemma 3

$b \notin A$	$b, a \in B$	$a \notin C$
$a, c \in A$	$c \notin B$	$c, d \in C$

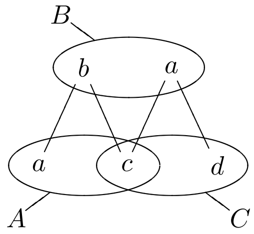

Figure 5. The Sets of Lemma 3

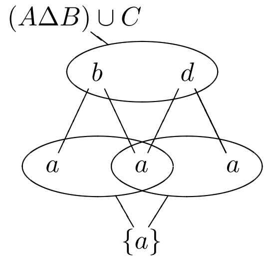

Figure 6. Resulting zigzag of Lemma 3

Lemma 3. *Let $\mathcal{F}$ be a PSF on S. Let also A, B, C be elements of $\mathcal{F}$ (see Figure 5) satisfying the conditions of Table 2. Then $(A\Delta B) \cup C \in \mathcal{F}$ avoids a and contains b and d, so a, b, a, d, a is an $\mathcal{F}$-zigzag (figure 6).*

proof. First note that A and B overlap, so $A\Delta B \in \mathcal{F}$. Furthermore we also have that $b \in A\Delta B$, $d \in C$ while $a \notin A\Delta B$. Now $A\Delta B$ and C have a common element c, so $(A\Delta B) \cup C \in \mathcal{F}$.

It now remains to be shown that Figure 6 represents an $\mathcal{F}$-zigzag. First since $b \notin A$ and $d \in C$ it follows that $b \notin \{a\}$ and $d \notin \{a\}$. Furthermore since a is neither in $A\Delta B$ nor in C we have that $a \notin (A\Delta B) \cup C$ and a, b, a, d, a is indeed an $\mathcal{F}$-zigzag. □

proof. **(of Theorem 7)** Let $X \in \mathcal{F}$, since $\mathcal{F}$ is a PSF so in particular a WPSF we have from Theorem 1 that X is a module of $\langle S; \rightleftharpoons_{\mathcal{F}} \rangle$, therefore also a module of $\langle S, \Leftrightarrow_{\mathcal{F}} \rangle$.

It remains now to be shown that if X is a module of $\langle S, \Leftrightarrow_{\mathcal{F}} \rangle$, then $X \in \mathcal{F}$. Using Proposition 1 as in the proof of Theorem 1, it is sufficient to find for any $x \notin X$ and $y, z \in X$ an $F_x \in \mathcal{F}$ such that $x \notin F_x$ and $y, z \in F_x$.

Since X is a module it is sufficient to show that if $(x, y) \Leftrightarrow_{\mathcal{F}} (x, z)$ then there exists an $F_x \in \mathcal{F}$ avoiding x but containing y and z. Since $(x, y) \Leftrightarrow_{\mathcal{F}} (x, z)$ we either have that $(x, y) \rightleftharpoons_{\mathcal{F}} (x, z)$ or $(x, y) \rightleftharpoons_{\mathcal{F}} (z, x)$. In the first case the claim follows from Lemma 2 as in the proof of Theorem 1. In the second case, using

Lemma 2, we have a length 4 zigzag from (x, y) to (z, x). This is the hypothesis of Lemma 3 with $a = x, b = y$ and $d = z$. Now by Lemma 3 we can take F_x to be $(A\Delta B) \cup C$.

Since by Proposition 6 the relation $\Leftrightarrow_{\mathcal{F}}$ is included in any symmetric $\equiv$ such that the family of modules of $\langle B, \equiv\rangle$ is $\mathcal{F}$, we now have that $\Leftrightarrow_{\mathcal{F}}$ is the finer such relation. □

11.1. **The logic of symmetric 2-structures.** The notion of symmetric 2-structure is obviously first-order since a 2-structure is symmetric exactly when it satisfies $\Delta = \forall x, y(x, y) \equiv (y, x)$.

Consider FO-S2S the *first-order theory of symmetric* 2*-structures*, which is the set of first-order sentences in the language $\{\equiv\}$ true in every symmetric 2-structure. Since the first-order theory of finite non-oriented graphs is undecidable [28, Theorem 4.2] using an argument similar to the one given at the beginning of section 5 to show the undecidability of FO-2S, we have that FO-S2S is undecidable. We therefore also have undecidability for the Monadic Second Order theory of symmetric 2-structures MSO-S2S.

Consider now the MSO theory of the structures $\langle B, \Leftrightarrow_{\mathcal{F}}\rangle$ for $\mathcal{F}$ a PSF. As in the case of WPSFs we have the following facts.

(1) The family of modules of $\langle S, \Leftrightarrow_{\mathcal{F}}\rangle$ is $\mathcal{F}$.
(2) The relation $\Leftrightarrow_{\mathcal{F}}$ is MSO-definable in any symmetric 2-structure whose set of modules is $\mathcal{F}$.
(3) The class of models $\mathcal{M}$(MSO-S2SF) of MSO-S2SF is MSO-definable.
(4) MSO-S2SF is decidable by interpretation in WMS2S.

The first fact follows from Theorem 7. The second fact is obvious since $\Leftrightarrow_{\mathcal{F}}$ is the symmetric closure of $\rightleftharpoons_{\mathcal{F}}$, which is as we have shown in Theorem 2, definable in any 2-structure whose set of modules is $\mathcal{F}$. For the third fact, MSO-S2SF is MSO-definable by the conjunction of the sentences Δ expressing that the structure is symmetric with a sentence expressing that $\equiv$ is equal to $\Leftrightarrow_{\mathcal{F}}$. This last sentence being similar to the one given in section 5 to show the MSO-definability of MSO-2SF.

For the last fact, note that a sentence $\psi \in$MSO-2SF if and only if $\Delta \rightarrow \psi \in$MSO-2SF. Since MSO-2SF, is decidable by interpretation in WMS2S, the same holds for MSO-S2SF.

11.2. **Number of classes.** For a symmetric 2-structure on a three element set, if one has only 2 equivalence classes there is always one vertex which is in two equivalent distinct pairs, therefore there is always a module containing 2-elements. We therefore have that in order to realize the trivial PSF on a three element set, one needs a symmetric 2-structure with 3 equivalence classes.

For completeness, let us state that with a construction similar to the one of section 10, one can realize any PSF by a symmetric labeled 2-structure of rank 3 with 3 equivalence classes. Furthermore since a PSF is closed under the symmetric difference of overlapping elements, its modular decomposition tree contains only trivial and complete nodes (the symmetric difference of two intervals not being necessarily an interval).

Finally the number of classes of $\Leftrightarrow_{\mathcal{F}}$ is easily computed since this relation is the symmetric closure of the equivalence relation of $\mathcal{L}_{\mathcal{F}}$'s underlying 2-structure. Since we add in $\mathcal{L}_{\mathcal{F}}$ for every trivial node P, with children $P_1, \ldots, P_n$, $n(n-1)$ new classes (one for each distinct pair of element of $\{P_1, \ldots, P_n\}$), we will have in this case $n(n-1)/2$ new classes in $\rightleftharpoons_{\mathcal{F}}$. If P is complete we add a single symmetric class to $\mathcal{L}_{\mathcal{F}}$, hence in that case $\rightleftharpoons_{\mathcal{F}}$ gets a single new class. We therefore have the following recurrence where P is a node of the modular decomposition tree whose children are $P_1, \ldots, P_n$.

$$no\_classes(P) = \begin{cases} n(n-1)/2 & + \sum_i no\_classes(P_i) \quad \text{for } P \text{ trivial} \\ 1 & + \sum_i no\_classes(P_i) \quad \text{for } P \text{ complete} \end{cases}$$

The total number of equivalence classes of $\rightleftharpoons_{\mathcal{F}}$ being $no\_classes(R)$ for R the root of $\mathcal{A}$.

12. Conclusion

We have shown that not only is the notion of family of modules axiomatized by the notion of WPSF, but furthermore that one can explicitly construct an equivalence relation from a WPSF by a MSO-formula. This has allowed us to MSO-define inside any 2-structure $\langle S, \equiv\rangle$ another 2-structure $\langle S, \rightleftharpoons_{\mathcal{F}}\rangle$ having the same family of modules $\mathcal{F}$. We have furthermore shown that the theory MSO-2SF of these structures is MSO-definable and is interpretable in the weak monadic Second Order theory of 2 successor WMS2S, which is well known to be decidable by automata. Finally we have given a recurrence to compute the number of classes of $\rightleftharpoons_{\mathcal{F}}$ and generalized these results to PSF.

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