Generalized trees related with tree metrics

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Abstract

Graphs without induced subgraphs of minimum degree at least 3 have been considered in matroid theory (see, e.g., Welsh 1976, ch. 14 or Aigner 1979, ch. 7) and in some domains of applications (Todd 1989, Guénoche and Leclerc 1999). Maximal such graphs were characterized in Todd's paper. They are called here 2*d-trees*.

A subclass of 2d-trees consists of the classical 2-trees (Pippert and Beineke 1969), whose interesting status among chordal, series parallel and 2-connected graphs has been extensively studied in the literature (Arnborg and Proskurowski 1989, Borie, Parker and Tovey 1991, Rose 1974, and others). We consider here valued 2d-trees and 2-trees with a given finite vertex set X.

An XLL-tree (*leaf labelled according to* X) is a tree $T = (V(T), E(T))$ satisfying two properties: (i) the leaf set of T is X; (ii) for any $v \in V(T)$-X, $\delta(v) \geq 3$ (where $\delta(v)$ is the degree of v). Elements of $V(T)$-X are the *latent vertices* of T. For definitions and properties about such trees, see Barthélemy and Guénoche (1988). Assume that an XLL-tree T is valued in such a way that any edge between two latent vertices is non-negative; then it defines a tree function on X, that is a real function t on the set $X^{(2)}$ of pairs of distinct elements of X satisfying the *weak four-point condition*: for all *distinct* $x, y, z, w \in X$, $t(xy) + t(zw) \leq \max\{t(xz) + t(yw), t(xw) + t(yz)\}$. For any distinct $x, y \in X$, $t(xy)$ is given by the sum of the edge values on the path of

T between x and y. Conversely, a unique valued XLL-tree T is associated to any given tree function on X (Buneman 1974).

A tree function t is a *tree dissimilarity* if it is positive, and a *tree metric* if, moreover, it corresponds with a non-negatively valued XLL-tree.

A basic fact is that any valued 2d-tree $\Theta = (X, E(\Theta))$ uniquely defines a valued XLL-tree or, equivalently, a tree function t on X (Leclerc and Makarenkov 1998, Guénoche and Leclerc 1999). Conversely, consider a tree function t on X as a valued complete graph K_X. Then, some restrictions of t to 2d-trees provide encodings of t by $2n$-3 values. This fact generalizes such encodings previously recognized in the literature (Chaiken et al. 1983, Yushmanov 1984). It has been recently applied to phylogenetic reconstruction from incomplete dissimilarity data. We study and characterize such restrictions. Those corresponding to 2-trees lead to several types of minimum spanning 2-trees, each with properties of the minimum spanning tree type.

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