Enumeration of eulerian and unicursal planar maps

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Received October 2002; received in revised form 30 April 2003; accepted September 2003

Dedicated to the memory of Professor William T. Tutte

Abstract

Sum-free enumerative formulae are derived for several classes of rooted planar maps with no vertices of odd valency (*eulerian* maps) and with two vertices of odd valency (*unicursal* maps). As corollaries we obtain simple formulae for the numbers of unrooted eulerian and unicursal planar maps. Also, we obtain a sum-free formula for the number of rooted *bi-eulerian* (eulerian and bipartite) maps and some related results.

Keywords: Rooted planar map; Unrooted eulerian map; Sum-free formula; Lagrange inversion; Bipartite

1 Introduction

1.1 Eulerian maps have played a crucial role in enumerative map theory since its beginning in the early sixties. In particular, Tutte's sum-free formula [22] for the number of eulerian planar maps, all of whose vertices are labelled and contain a distinguished edge-end, with a given sequence of (even) vertex valencies was an essential step in obtaining his ground-breaking formula for counting rooted planar maps by number of edges [23]. Several new results on the subject have been published since then (see, e.g., [24, 9, 19, 2, 14, 4, 17]).

Here we consider two types of planar maps: eulerian maps - maps with no vertices of odd valency - and unicursal maps - maps with exactly two vertices of odd valency. It turns out that in most cases under consideration the rooted maps are counted by sum-free formulae. Such formulae are both elegant and computationally efficient; they facilitate investigating asymptotic behaviour and various arithmetic properties. Very often sum-free formulae enlarge the enumerative role of the corresponding objects. Generally, it is difficult to predict such formulae; so it is always a pleasant surprise to discover them.

A sum-free formula for the number of rooted eulerian planar maps with a given number of edges n appears in [24] (in [23, p. 269] the same formula counts rooted bipartite trivalent (bicubic) maps with $3n$ edges, and a bijection between these two classes of maps was first presented in [15]). In Section 2 we find sum-free formulae for rooted unicursal planar maps with a given number of edges and for those with zero, one or two endpoints. We also find a sum-free formula for the number of unicursal maps rooted in a vertex of odd valency and a formula for the number of rooted unicursal maps as a function of the odd vertex valencies.

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‡ Partially supported by the INTAS (Grant INTAS-BELARUS 97-0093).

§ Supported by National Science and Engineering Research Council (NSERC) under grant RGPIN-44703.

In Section 3 we apply the methods of [9], and a formula obtained therein, to the results of Section 2 to count unrooted eulerian and unicursal maps by number of edges.

In Section 4 we obtain (based on [7]) a sum-free formula for the number of rooted bi-eulerian (eulerian bipartite) maps. We also obtain formulae (much less elegant) to count those rooted eulerian and bi-eulerian maps that are non-separable.

Finally, in Section 5, we present some asymptotic formulae, discuss some identities and pose several open problems.

1.2 Basic definitions. A *map* means a planar map: a 2-cell embedding of a planar connected graph (loops and multiple edges allowed) in an oriented sphere. A map is *rooted* if one of its edge-ends (variously known as edge-vertex incidence pairs, darts, semi-edges, or "brins" in French) is distinguished as the *root*. Counting *unrooted* maps means counting maps up to orientation-preserving homeomorphism.

A map (or a graph) is *eulerian* if it has an eulerian circuit - that is, a circuit containing each of the edges exactly once. It is well-known that a map (or connected graph) is eulerian if and only if all its vertices are of even valency. A map (or graph) is *bipartite* if its vertices can be partitioned into two parts so that no two vertices in the same part are connected by an edge. It is also well known that a map is bipartite if and only if all its faces are of even valency. Thus for planar maps, these two properties - eulerian and bipartite - are related by face-vertex duality. A map which is both eulerian and bipartite is called *bi-eulerian*.

A map or graph is generally called unicursal if it possesses an eulerian walk, not necessarily a circuit. It is well known that a map (or connected graph) is unicursal if and only if it contains no more than two vertices of odd valency. For the sake of brevity we abuse the term and call a map *unicursal* if it has *exactly* two vertices of odd valency. An *endpoint* is a vertex of valency 1; a unicursal map evidently can have at most two such vertices.

Finally, a map is called *non-separable* if its edge-set cannot be partitioned into two non-empty parts such that only one vertex and one face incident with it are incident with an edge in each part. A planar map with at least two edges is non-separable if and only if it has no loops and its underlying graph is 2-connected [23].

2 Rooted unicursal maps

Unicursal maps and eulerian maps are the very maps considered by Tutte in his seminal paper [22]. But all the enumerative results obtained so far for unicursal maps concern maps with specified vertex valencies. Accordingly, up until now no formula has been known for the number of rooted unicursal maps with n edges. As we mentioned above, a formula is known for the number of eulerian maps with n edges. However the problem of counting unicursal maps cannot be reduced to the analogous problem for eulerian maps by adding an edge connecting the two odd-valent vertices because these vertices may not be incident to a common face; so we have to consider unicursal maps independently.

2.1 Let $U'(n)$ denote the number of rooted unicursal maps with n edges and let $U'_i(n),\ i = 0, 1, 2,$ denote the number of rooted unicursal maps with i endpoints.

Theorem 1.

$$U'(n) = 2^{n-2}\binom{2n}{n}, \qquad n \geq 1, \tag{2.1}$$

and for $n \geq 2$,

$$U'_0(n) = 2^{n-2}\,\frac{n-2}{n}\binom{2n-2}{n-1}, \tag{2.2}$$

$$U'_1(n) = 2^{n-1}\binom{2n-2}{n-1} \tag{2.3}$$

and

$$U_2'(n) = 2^{n-2}\binom{2n-2}{n-1}. \tag{2.4}$$

Proof. The number of unicursal planar maps with n edges and v vertices labelled $1, 2, \ldots, v$ with the vertex i of valency $2d_i + 1$ if $i = 1, 2$ and $2d_i$ if $i > 2$ and each vertex rooted by distinguishing one of its edge-ends, is given in [22, p. 772] as

$$C(2d_1+1, 2d_2+1, 2d_3, \ldots, 2d_v) = \frac{(n-1)!}{(n-v+2)!}\frac{(2d_1+1)!(2d_2+1)!}{d_1!^2 d_2!^2}\prod_{i=3}^{v}\frac{(2d_i)!}{d_i!(d_i-1)!}. \tag{2.5}$$

The number of rooted planar maps with n edges and v vertices, exactly two of which are of odd valency, is found from the previous equation by multiplying by the number of ways to root a map with n edges and dividing by the number of ways to label and root all the vertices of the same map so that the two vertices of odd valency get labels 1 and 2 (we multiply by $2n$ and divide by the product of the valencies and by $v!$ and then multiply by $v(v-1)/2$ to account for the fact that the two vertices of odd valency get labels 1 and 2) and then summing over the sequences of valencies that add to $2n$:

$$\frac{n!}{(v-2)!(n-v+2)!}\sum_{d_1+\cdots+d_v=n-1}\left\{\frac{(2d_1)!(2d_2)!}{d_1!^2 d_2!^2}\prod_{i=3}^{v}\frac{(2d_i-1)!}{d_i!(d_i-1)!}\right\}.$$

To obtain $U'(n)$ we evaluate the sum and then add over all possible values of v : from 2 to $n+1$.

Since $\sum_{j=0}^{\infty}\frac{(2j)!}{j!^2}x^j = (1-4x)^{-1/2}$ and $\sum_{j=1}^{\infty}\frac{(2j-1)!}{j!(j-1)!}x^j = \frac{(1-4x)^{-1/2}-1}{2}$, we have

$$U'(n) = [x^{n-1}]\sum_{v=2}^{n+1}\frac{n!}{(v-2)!(n-v+2)!}(1-4x)^{-1}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-2}, \tag{2.6}$$

where $[x^n]b$ means the coefficient of x^n in the power series $b = b(x)$.

We set $z := x(z+1)^2$, so that $\frac{(1-4x)^{-1/2}-1}{2} = \frac{z}{1-z}$ and $(1-4x)^{-1} = \left(\frac{1+z}{1-z}\right)^2$. Then

$$\begin{aligned}
U'(n) &= [x^{n-1}]\sum_{v=2}^{n+1}\binom{n}{v-2}\left(\frac{1+z}{1-z}\right)^2\left(\frac{z}{1-z}\right)^{v-2} \\
&= [x^{n-1}]\left(\frac{1+z}{1-z}\right)^2\sum_{v=0}^{n-1}\binom{n}{v}\left(\frac{z}{1-z}\right)^{v} \\
&= [x^{n-1}]\left(\frac{1+z}{1-z}\right)^2\left[\left(1+\frac{z}{1-z}\right)^n - \left(\frac{z}{1-z}\right)^n\right] \\
&= [x^{n-1}]\Big[(1+z)^2(1-z)^{-(n+2)} - (1+z)^2 z^n (1-z)^{-(n+2)}\Big].
\end{aligned}$$

By Lagrange's inversion formula (see, e.g., [8]),

$$U'(n) = \frac{1}{n-1}[z^{n-2}]\left\{(1+z)^{2n-2}\frac{d}{dz}\Big[(1+z)^2(1-z)^{-(n+2)} - (1+z)^2 z^n(1-z)^{-(n+2)}\Big]\right\}.$$

Now a factor of z^n means that the coefficient of z^{n-2} will be zero even in the derivative. We have

$\frac{d}{dz}\left[(1+z)^2(1-z)^{-(n+2)}\right] = 2(1+z)(1-z)^{-(n+2)} + (n+2)(1+z)^2(1-z)^{-(n+3)}$, so that

$$\begin{aligned}
U'(n) &= \frac{1}{n-1}[z^{n-2}]\left[2(1+z)^{2n-1}(1-z)^{-(n+2)} + (n+2)(1+z)^{2n}(1-z)^{-(n+3)}\right] \\
&= \frac{1}{n-1}\left[2\sum_{i=0}^{n-2}\binom{2n-1}{n-2-i}\binom{i+n+1}{i} + (n+2)\sum_{i=0}^{n-2}\binom{2n}{n-2-i}\binom{i+n+2}{i}\right] \\
&= \frac{1}{n-1}\left[2\frac{(2n-1)!}{(n+1)!(n-2)!}\sum_{i=0}^{n-2}\binom{n-2}{i} + (n+2)\frac{(2n)!}{(n+2)!(n-2)!}\sum_{i=0}^{n-2}\binom{n-2}{i}\right] \\
&= \frac{2^{n-2}}{n-1}\left[2\frac{(2n-1)!}{(n+1)!(n-2)!} + \frac{(2n)!}{(n+1)!(n-2)!}\right],
\end{aligned}$$

which simplifies to (2.1). This derivation is valid only for $n \geq 2$ since we are taking coefficients of z^{n-2}, but (2.1) turns out to be valid for $n = 1$ as well.

To prove formula (2.4) we set d_1 and d_2 to 0, so that the first and second vertices have valency 1. Proceeding as above, we find that

$$U_2'(n) = [x^{n-1}]\sum_{v=2}^{n+1}\frac{n!}{(v-2)!(n-v+2)!}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-2},$$

so that, by Lagrange's inversion formula,

$$U_2'(n) = \frac{1}{n-1}[z^{n-2}]\left\{(1+z)^{2n-2}\frac{d}{dz}\left[(1-z)^{-n} - z^n(1-z)^{-n}\right]\right\},$$

which simplifies to (2.4) using the same type of calculation as above.

If instead we just set d_1 to 0, then the first vertex has valency 1 and the second vertex can have any odd valency $2d_2+1$, including 1. If $d_2 = 0$ then, as before, we multiply by $n(n-1)/2$ to account for the fact that the two vertices of valency 1 get labels 1 and 2, but if $d_2 > 0$, then we instead multiply by $n(n-1)$ to account for the fact the vertex of valency 1 gets label 1 and the other odd-valent vertex gets label 2; so

$$2U_2'(n) + U_1'(n) = 2[x^{n-1}]\sum_{v=2}^{n+1}\frac{n!}{(v-2)!(n-v+2)!}(1-4x)^{-1/2}\left[\frac{(1-4x)^{-1/2}-1}{2}\right]^{v-2}. \tag{2.7}$$

Therefore by Lagrange's inversion formula,

$$2U_2'(n) + U_1'(n) = \frac{1}{n-1}[z^{n-2}]\left\{(1+z)^{2n-2}\frac{d}{dz}\left[(1+z)(1-z)^{-(n+1)} - (1+z)z^n(1-z)^{-(n+1)}\right]\right\}.$$

This formula simplifies, in the same manner as above, to

$$2^n\binom{2n-2}{n-1},$$

from which (2.3) follows.

Finally, formula (2.2) follows from the other formulae since

$$U_0'(n) + U_1'(n) + U_2'(n) = U'(n).$$

Formulae (2.2), (2.3) and (2.4) are valid only for $n \geq 2$. □

Remark. W. Tutte did not publish a proof of (2.5) because, as he informed one of the authors (Walsh) in late 2001, he had not expected that formula to have any applications (and he expressed satisfaction upon hearing about the use to which we have put it). A bijective proof of both (2.5) and the corresponding formula for eulerian maps, which was proved in [22], appeared in [5]. Another bijective proof of the latter formula appeared in [18]. The method used in [18] was generalized in [3] to count rooted maps with 1 or 2 endpoints.

2.2 Similar calculations yield sum-free formulae for rooted unicursal maps satisfying various conditions. For example, to find the number $\widetilde{U}'(n)$ of unicursal maps with n edges *rooted at an odd-valency vertex*, we do the same calculation, beginning with the formula for $C(2d_1+1, 2d_2+1, 2d_3, \dots, 2d_v)$, except that instead of multiplying by $2n$ we multiply by $2(d_1+d_2+1)$. We spare the reader the tedious details and give the final result:

Theorem 2.

$$\widetilde{U}'(n) = \frac{2^{n-2}}{n+2}\binom{2n+2}{n+1}. \tag{2.8}$$

Similarly, to find the number $U'(2d_1+1, 2d_2+1; n)$ of rooted unicursal maps with n edges and two vertices of fixed odd valencies $2d_1+1$ and $2d_2+1$, $d_1 \le d_2$, instead of taking the sum over $d_1 + \dots + d_v = n-1$ we take it over $d_3 + \dots + d_v = n - d_1 - d_2 - 1$ (and we multiply this sum by 2 if $d_1 \ne d_2$). The same procedure leads to the product of two factors F_1 and F_2, where

$$F_1 = \binom{2d_1}{d_1}\binom{2d_2}{d_2}\frac{(2n-2d_1-2d_2-2)!}{(n-1)!(n-d_1-d_2-1)!} \times \begin{cases} 2 & \text{if} \quad d_2 \ne d_1 \\ 1 & \text{if} \quad d_2 = d_1 \end{cases}$$

and with $d = d_1 + d_2$,

$$F_2 = \sum_{i=0}^{n-d-2} \frac{(n+i)!}{(n-d+i)!}\binom{n-d-2}{i}.$$

Using the identity

$$\frac{(n+i)!}{(n+i-d)!} = \sum_{j=0}^{d}\binom{d}{j}\frac{n!}{(n-j)!}\frac{i!}{(i-d+j)!},$$

which can be proved, e.g., by induction on d, we show that

$$F_2 = (n-d-2)! \sum_{j=0}^{\min(d,n-d-2)} \binom{d}{j}\frac{n!}{(n-d+j)!(n-d-2-j)!}\, 2^{n-d-2-j}.$$

Thus, we prove the following.

Theorem 3. *For all* $n \ge d+2$,

$$U'(2d_1+1, 2d_2+1; n) = \frac{n\binom{2d_1}{d_1}\binom{2d_2}{d_2}}{n-d-1} \sum_{j=0}^{\min(d,n-d-2)} \binom{d}{j}\binom{2n-2d-2}{n-d-2-j} 2^{n-d-2-j} \times \begin{cases} 2 & \text{if } d_2 \ne d_1 \\ 1 & \text{if } d_2 = d_1 \end{cases} \tag{2.9}$$

where $d = d_1 + d_2$ *and* $U'(2d_1+1, 2d_2+1; n)$ *is the number of rooted unicursal maps with* n *edges and two vertices of odd valencies* $2d_1+1$ *and* $2d_2+1$.

If $n = d+1$ (the smallest value n can have), then $v = 2$; so from (2.5) we have: $U'(2d_1+1, 2d_2+1; d+1) = 2\binom{2d_1}{d_1}\binom{2d_2}{d_2}$ if $d_2 \ne d_1$ and $U'(2d_1+1, 2d_1+1; 2d_1+1) = \binom{2d_1}{d_1}^2$.

For small d_1 and d_2, the right-hand side of (2.9) can be made sum-free. In particular $U'(1,1;n) = U_2'(n)$ and (2.9) reduces to (2.4). If the odd valencies are 1 and 3, then formula (2.9) simplifies to

$$U'(1,3;n) = 3 \cdot 2^{n-2}\binom{2n-4}{n-2}, \quad n \ge 3, \tag{2.10}$$

(moreover, $U'(1,3;2) = 4$). It simplifies to

$$U'(3,3;n) = \frac{9n-20}{n-2}\, 2^{n-4}\binom{2n-6}{n-3}, \quad n \ge 4, \tag{2.11}$$

($U'(3,3;3) = 4$) and to three times that number for $U'(1,5;n)$.

In general, for arbitrary fixed d_1 and d_2 formula (2.9) after elementary transformations can be represented as follows:

Corollary. *For* $d = d_1 + d_2$ *and all* $n \geq d + 2$,

$$U'(2d_1+1, 2d_2+1; n) = \frac{h_d(n)\, 2^{n-2d-2}}{(n-2)(n-3)\cdots(n-d)} \binom{2d_1}{d_1} \binom{2d_2}{d_2} \binom{2n-2d-2}{n-d-1} \times \begin{cases} 2 & \text{if } d_2 \neq d_1 \\ 1 & \text{if } d_2 = d_1 \end{cases} \tag{2.9'}$$

where $h_0(n) = 1$ *and for* $d \geq 1$, $h_d(n)$ *is the following polynomial of* n *of degree* $d - 1$:

$$h_d(n) = \frac{1}{n-1} \sum_{j=0}^{d} \binom{d}{j} 2^{d-j} (n-d-2)(n-d-3)\cdots(n-d-1-j) \cdot n(n-1)\cdots(n-d+j+1). \tag{2.12}$$

It is easy to see that the sum in (2.12) is divisible by $n - 1$. Now, by (2.12),

$h_1(n) = 3$ (this is formula (2.10)),

$h_2(n) = 9n - 20$ (cf. (2.11)),

$h_3(n) = 3(3n - 7)(3n - 10)$,

$h_4(n) = 3(27n^3 - 279n^2 + 934n - 1008)$, etc.

2.3 Rooted eulerian maps. The number $E'(n)$ of rooted eulerian planar maps with n edges is expressed by the following well-known formula [24]:

$$E'(n) = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}. \tag{2.13}$$

Denoting $e(x) := \sum_{n=0}^{\infty} E'(n) x^n$, it can be easily verified that

$$e(x) = \frac{8x^2 + 12x - 1 + (1 - 8x)^{3/2}}{32x^2}. \tag{2.14}$$

3 Unrooted eulerian and unicursal maps

3.1 Formulae (2.1), (2.3), (2.4) and (2.13) enable us to complete the solution of the long-standing problem of the enumeration of unrooted eulerian planar maps. Namely, a formula obtained in [9] can be transformed into an explicit formula with single sums over the divisors of n.

Theorem 4. *The number* $E^+(n)$ *of non-isomorphic eulerian planar maps with* n *edges,* $n \geq 2$, *is expressed as follows:*

$$\begin{aligned} E^+(n) &= \frac{1}{2n} \Bigg[\frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n} + 3 \sum_{k<n,\, k|n} \phi\Big(\frac{n}{k}\Big) 2^{k-2} \binom{2k}{k} \\ &+ \begin{cases} \dfrac{n \cdot 2^{(n+1)/2}}{n+1} \dbinom{n-1}{\frac{n-1}{2}}, & n \text{ odd}, \\ \displaystyle\sum_{k|\frac{n}{2}} \phi\Big(\frac{n}{k}\Big) 2^{k-2} \binom{2k}{k} + \frac{n \cdot 2^{(n-2)/2}}{n+2} \binom{n}{\frac{n}{2}}, & n \text{ even}, \end{cases} \Bigg] \end{aligned} \tag{3.1}$$

where $\phi(n)$ *is the Euler totient function.*

Proof. This expression is an easy consequence of the following result.

Theorem 5 [9]**.**

$$\begin{aligned} E^+(n) &= \frac{1}{2n} \Bigg[E'(n) + \frac{1}{2} \sum_{k<n,\, k|n} \phi\Big(\frac{n}{k}\Big) (k+2)(k+1) E'(k) \\ &+ \begin{cases} U'_*\Big(\dfrac{n+1}{2}\Big), & n \text{ odd}, \\ \displaystyle\sum_{k|\frac{n}{2}} \phi\Big(\frac{n}{k}\Big) U'(k) + U'_{**}\Big(\frac{n+2}{2}\Big), & n \text{ even}, \end{cases} \Bigg] \end{aligned} \tag{3.2}$$

where $U'_*(n)$ and $U'_{**}(n)$ denote the numbers of rooted unicursal maps with n edges and with one and two singular vertices respectively; a singular *vertex means an endpoint which is not allowed to be the root-vertex.*

It is clear that

$$U'_{**}(n) = \frac{n-1}{n}\, U'_2(n) \tag{3.3}$$

since any map with n edges and two singular vertices contains $2n$ edge-ends, of which exactly two are ineligible to be the root.

Likewise

$$U'_*(n) = \frac{2n-1}{2n}\, U'_1(n) + \frac{2n-1}{n}\, U'_2(n).$$

Indeed, the first summand reflects the fact that we may take any unrooted unicursal map with a unique endpoint, declare this vertex to be singular and choose a root in one of $2n-1$ ways. The second summand is obtained by considering the contribution to the set of rooted maps with one singular vertex made by a map Γ with two endpoints. If Γ has no non-trivial symmetries, then we must declare one of its endpoints to be singular and then choose a root in one of $2n-1$ ways; so Γ contributes $2(2n-1)$ to the set of rooted maps with one singular vertex instead of the usual $2n$ rootings. Now suppose that Γ has a rotational symmetry of order 2 (the only possible non-trivial orientation-preserving automorphism). Both endpoints are equivalent, and after we declare one of them to be singular (which destroys the symmetry), there are $2n-1$ (instead of n) possible rootings. Therefore, in both cases the proportion $(2n-1) : n$ is the same (alternatively, this ratio could be obtained by using doubly rooted maps).

Finally, taking into account formulae (2.3) and (2.4) we obtain

$$U'_*(n) = \frac{2n-1}{n}\, U'_1(n). \tag{3.4}$$

Substituting from (2.13), (2.1), (3.3), (3.4), (2.4) and (2.3) into (3.2) we obtain (3.1). □

3.2 Similarly we prove the following.

Theorem 6. *Let $U^+(n)$ denote the number of non-isomorphic unicursal planar maps with n edges, $n \geq 2$, then*

$$U^+(n) = \frac{1}{2n} \sum_{\substack{k|n \\ n/k \text{ odd}}} \phi\Big(\frac{n}{k}\Big) 2^{k-2}\binom{2k}{k} + \begin{cases} 2^{(n-3)/2}\dbinom{n-1}{\frac{n-1}{2}}, & n \text{ odd}, \\ 2^{(n-6)/2}\dbinom{n}{\frac{n}{2}}, & n \text{ even}. \end{cases} \tag{3.5}$$

Proof. We exploit the method developed in [9]. Unicursal maps are similar to but simpler than eulerian maps with respect to possible rotational symmetries. Namely, only three following types of rotations exist (in the mnemonic designations adopted in [9]):

(I_ℓ) rotations of an odd order $\ell \geq 3\,(\ell k = n)$ around the two odd-valent vertices (that is, around an axis that intersects the map in the two odd-valent vertices);

(I_2) rotations of order 2 around two even-valent vertices or an even-valent vertex and (the center of) a face;

(T) rotations of order 2 around the middle of an edge and a vertex or a face.

In every case, the quotient map is a unicursal map; it contains one singular vertex in the last case.

Now consider the possible liftings. In the first case, the axial cells (the vertices, edges or faces in which the axis of rotation intersects the map) are determined uniquely. For I_2 we choose one odd-valent vertex of the quotient map as axial; the other axial cell is an arbitrary vertex or face except for the second odd-valent vertex. These are the possible choices of the second axial vertex for rotations of the type T as well, while the first axial cell is necessarily the singular vertex. Now, by the main theorem of [9] we obtain immediately the formula

$$U^+(n) = \frac{1}{2n}\left[\sum_{\substack{k|n \\ n/k \text{ odd}}} \phi\Big(\frac{n}{k}\Big) U'(k) + \begin{cases} nU'\Big(\dfrac{n}{2}\Big), & n \text{ odd}, \\ \dfrac{n+1}{2} U'_*\Big(\dfrac{n+1}{2}\Big), & n \text{ even}. \end{cases} \right]$$

This formula, together with (2.1), (2.3) and (3.4), gives rise to (3.5). □

The unicursal maps with at most three edges are depicted in Fig. 1, where below every map we indicate the number of ways of rooting it.

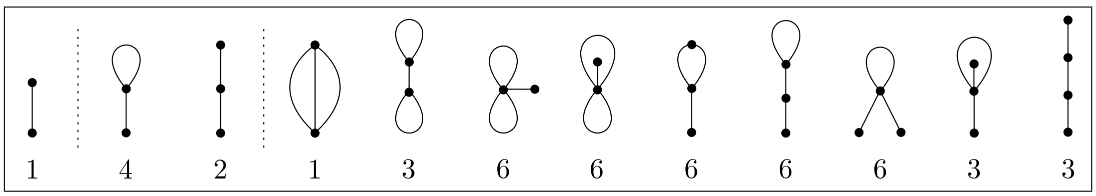

Figure 1: The unicursal planar maps with ≤ 3 edges

3.3 Specializing this proof to unrooted unicursal maps with two endpoints we obtain the following expression for their number $U_2^+(n)$.

Proposition 1. *$U_2^+(1) = U_2^+(2) = 1$ and for $n \geq 3$,*

$$U_2^+(n) = \frac{1}{n} 2^{n-3} \binom{2n-2}{n-1} + 2^{m-3} \binom{2m-2}{m-1} \tag{3.6}$$

where $m = \lfloor (n+1)/2 \rfloor$.

The first term in this formula can be written as $2^{n-3} C_{n-1}$, where C_n is the n-th Catalan number. Notice also that unicursal maps with exactly one endpoint do not have non-trivial symmetries; therefore the number of such unrooted maps $U_1^+(n) = U_1'(n)/(2n)$, and by (2.3) we have

$$U_1^+(n) = 2^{n-2} C_{n-1}, \quad n \geq 2. \tag{3.7}$$

Likewise for unrooted unicursal maps with vertices of valencies 1 and 3, we have $U^+(1,3;n) = U'(1,3,n)/(2n)$, whence by (2.10) we obtain

$$U^+(1,3;n) = \frac{3}{n} 2^{n-3} \binom{2n-4}{n-2}, \quad n \geq 3. \tag{3.8}$$

4 Rooted bipartite and non-separable eulerian maps

4.1 Bi-eulerian maps. As we noticed in Sect. 1.2, the dual of an eulerian planar map is bipartite, and vice versa (thus $E'(n)$ expresses also the number of rooted bipartite maps). Moreover, in a bi-eulerian map, the edges of an eulerian circuit switch alternately between the two parts. Therefore the following assertion is valid.

Lemma. *A planar map is bi-eulerian if and only if all its vertices and faces are even-valent. Any bi-eulerian map contains an even number of edges.*

4.2 According to [7] (see also [21]), the cubic equation

$$3x^2y^3 - y + 1 = 0 \tag{4.1}$$

and

$$f(x) = (1 + 3y - y^2)/3 \tag{4.2}$$

determine the generating function $f(x) = 1 + \sum_{n=1}^{\infty} B'(2n)x^{2n}$ of the number $B'(2n)$ of rooted bipartite eulerian planar maps with $2n$ edges. This remarkable result has been obtained by a strong physical method known as the method of matrix integrals (see [26]) with the help of character expansion techniques.

From (4.1) and (4.2) one can easily obtain the following explicit sum-free formula:

Proposition 2.

$$B'(2n) = \frac{3^{n-1}}{n(2n+1)} \binom{3n}{n+1}. \tag{4.3}$$

Proof. Represent (4.1) in the form $w = 3x^2(w+1)^3$, where $w = y - 1$. Then (4.2) becomes $f(x) = (3 + w - w^2)/3$. Applying Lagrange's inversion formula, we obtain

$$[x^{2n}]\, f(x) = \frac{1}{n}[w^{n-1}]f'(w)(3(1+w)^3)^n = \frac{1}{n}[w^{n-1}]\frac{1-2w}{3}3^n(1+w)^{3n} = \frac{3^{n-1}}{n}\left\{\binom{3n}{n-1} - 2\binom{3n}{n-2}\right\},$$

which gives rise to (4.3). □

This is apparently a new result (announced in [12]), although as we learned not long ago [20, 16], D. Poulalhon and G. Schaeffer deduced formula (4.3) directly, based on the combinatorial technique developed in [2].

Remarks. 1. There is a simple 1:2:3 correspondence between, resp., rooted bi-eulerian planar maps with $2n$ edges, tetravalent bi-eulerian maps with $4n$ edges and trivalent maps with all face sizes multiple to 3 and with $6n$ edges. This claim has been established by Szabo and Wheater [21].

2. It is an easy matter to prove that bi-eulerian maps form a *degenerate* class of maps in the sense that they cannot be 3-connected (that is, polyhedral). Indeed, in a 3-connected map, all the vertices and faces are of valency at least 3; in a bi-eulerian map the valencies are all even, and so they must all be at least 4. Neither the number of vertices nor the number of faces can, therefore, exceed half the number of edges; plugging these two inequalities into the Euler formula yields a contradiction if the map is to be finite and planar. At the same time, there is an infinite 3-connected bi-eulerian planar map - the infinite chessboard - and lots of finite bi-eulerian maps of higher genus, the smallest of which (on the torus) has one vertex and one face, each of valency 4, and two edges.

4.3 Non-separable eulerian and bi-eulerian maps. The functional equation

$$f(x) = g(x(f(x))^2) \tag{4.4}$$

relates the generating function $f(x)$ that counts rooted maps by number of edges and the generating function $g(x)$ that counts rooted non-separable maps by number of edges [23, 25] (both $f(x)$ and $g(x)$ also count the map with one vertex and no edges). It holds as well if $f(x)$ counts a class $\mathfrak{M}$ of rooted maps and $g(x)$ counts the subclass of rooted non-separable maps in $\mathfrak{M}$ provided that a map is in $\mathfrak{M}$ if and only if all its 2-connected components are also in $\mathfrak{M}$. This condition holds for both eulerian maps and bi-eulerian maps; to prove this assertion, it suffices to delete the end components of the block-cutpoint tree and proceed by induction on the number of edges.

For eulerian maps, the function $f(x)$ is given in (2.14); for convenience we rewrite it here:

$$f(x) = \frac{8x^2 + 12x - 1 + (1-8x)^{3/2}}{32x^2}. \tag{4.5}$$

To apply (4.4) we express (4.5) in parametric form. Let

$$z = 2x(z+1)^2. \tag{4.6}$$

Solving (4.6) for x, substituting into (4.5) and simplifying, we find that

$$f(x) = 1 + \frac{z}{2} - \frac{z^2}{4}. \tag{4.7}$$

Now we set

$$u = x(f(x))^2. \tag{4.8}$$

From (4.4), (4.7) and (4.8) we find that

$$g(u) = 1 + \frac{z}{2} - \frac{z^2}{4}. \tag{4.9}$$

Eliminating x between (4.6) and (4.8) and substituting for $f(x)$ we obtain from (4.7) a relation between u and z which we express in a form suitable for Lagrange inversion:

$$z = \frac{2u(1+z)^2}{\left(1 + \frac{z}{2} - \frac{z^2}{4}\right)^2}. \tag{4.10}$$

The number $E'_{\mathrm{NS}}(n)$ of rooted non-separable eulerian maps with n edges is the coefficient of u^n in $g(u)$. Combining (4.9) and (4.10) and using Lagrange inversion, we express this number as

$$\frac{1}{n} \times [z^{n-1}] \frac{2^n(1+z)^{2n}}{\left(1 + \frac{z}{2} - \frac{z^2}{4}\right)^{2n}} \left(\frac{1-z}{2}\right). \tag{4.11}$$

Unfortunately, unlike the cases considered in Section 2, this coefficient cannot be expressed as a sum-free formula; in fact, it contains inconvenient alternating double sums:

$$E'_{\mathrm{NS}}(n) = \frac{1}{n} \times \left\{ \begin{array}{l} \sum_{j=0}^{n-1} (-1)^j \binom{2n+j-1}{j} \sum_{k=0}^{\min(j,n-j-1)} (-1)^k 2^{n-j-k-1} \binom{j}{k} \binom{2n}{n-j-k-1} \\ - \sum_{j=0}^{n-2} (-1)^j \binom{2n+j-1}{j} \sum_{k=0}^{\min(j,n-j-2)} (-1)^k 2^{n-j-k-2} \binom{j}{k} \binom{2n}{n-j-k-2} \end{array} \right\}. \tag{4.12}$$

Similarly for bi-eulerian maps we use the equations (4.1) and (4.2) which we rewrite as

$$f(x) = (1 + 3y - y^2)/2 \quad \text{and} \quad y = 1 + 3xy^3. \tag{4.13}$$

Setting $z = y - 1$ and applying the same procedure as before, we find that the number $B'_{\mathrm{NS}}(n)$ of rooted non-separable bi-eulerian maps with n edges is

$$B'_{\mathrm{NS}}(n) = \frac{1}{n} \times \left\{ \begin{array}{l} \sum_{j=0}^{n-1} (-1)^j 3^{n-j-1} \binom{2n+j-1}{j} \sum_{k=0}^{\min(j,n-j-1)} (-1)^k \binom{j}{k} \binom{3n}{n-j-k-1} \\ -2 \sum_{j=0}^{n-2} (-1)^j 3^{n-j-1} \binom{2n+j-1}{j} \sum_{k=0}^{\min(j,n-j-2)} (-1)^k \binom{j}{k} \binom{3n}{n-j-k-2} \end{array} \right\}. \tag{4.14}$$

These results appear to be new (cf. [14, Sect.6.2]).

5 Asymptotics, identities and open questions

5.1 Asymptotics. As direct corollaries of the obtained sum-free formulae we can obtain asymptotic estimates of the corresponding quantities. In particular, as $n \to \infty$,

$$U'(n) \sim \tfrac{1}{2\sqrt{\pi}}\, n^{-1/2} 8^n \tag{5.1}$$

and

$$B'(2n) \sim \tfrac{2\sqrt{6}}{3\sqrt{\pi}}\, (2n)^{-5/2} (9/2)^{2n}. \tag{5.2}$$

Formula (5.2) agrees with the well-known general cardinality pattern for planar maps [1] (cf. also [11])

$$C n^{-5/2} \rho^n, \qquad n \to \infty. \tag{5.3}$$

Here ρ is known as the *connective constant* (called also the growth constant), which depends on the class of maps, $\gamma = -5/2$ as the (universal) *critical exponent* and C as a multiplicative constant ($C\sqrt{\pi}$ is usually algebraic). More accurately, this behaviour is assumed to hold for $n \in \mathrm{Dom}$, where

Dom $\subseteq \mathbb{N}$ denotes the set of all n for which there exist n-edged maps of the class under consideration. Thus in (5.2), Dom $= 2\mathbb{N}$ and $\rho = 9/2$.

Unlike (5.2), in (5.1) we see another critical exponent: $\gamma = -1/2$. However this deviation from the pattern (5.3) is not counter-intuitive because in unicursal maps two vertices are distinguished implicitly. This introduces the additional factor $\binom{v}{2}$, which is of order n^2 since in the majority of maps, the number v of vertices is of order n. As a matter of fact, formula (5.1) satisfies a *generalized* planar map cardinality pattern which will be considered elsewhere (see [13]).

Non-separable eulerian and bi-eulerian maps satisfy the pattern (5.3). This assertion can be proved by the technique described in [1] (Darboux's method). We restrict ourselves to the calculation of ρ in order to supplement the table of known values of the connective constants given in [10].

For bi-eulerian maps, consider again the cubic equation (4.1), $3x^2y^3 - y + 1 = 0$. According to [1], the radius of convergence of y as a function of x is defined by the equation $81x^2 = 4$. Now we need to solve (4.1) at $x = 2/9$ and then evaluate the generating function $f(x) = (1+3y-y^2)/3$. The roots are $-3, 3/2$ and $3/2$. The root -3 is meaningless, and we have $f(2/9) = (1+3y-y^2)/3|_{y=3/2} = 13/12$. Finally, the radius of convergence of the generating function $f(x)$ for non-separable bi-eulerian maps is equal to $xf(x)^2|_{x=2/9} = 2/9\cdot(13/12)^2 = 169/648$. Therefore the connective constant $\rho = 648/169$.

Similarly for E', the radius of convergence of the function $f(x)$ (see (4.5)) is equal to $1/8$ and $f(1/8) = 5/4$. Thus the radius of convergence of the generating function $f(x)$ for non-separable eulerian maps is equal to $xf(x)^2|_{x=1/8} = 1/8 \cdot (5/4)^2 = 25/128$. Therefore here we have the connective constant $\rho = 128/25$.

As for unrooted maps, from (3.1) and (3.5) we obtain immediately that

$$E^+(n) \sim E'(n)/2n$$

and

$$U^+(n) \sim U'(n)/2n$$

as $n \to \infty$. In other words, "almost all" eulerian and unicursal maps have only the trivial symmetry.

5.2 Identities. Formulae (2.3) and (2.4) imply the following formal identity:

$$U_1'(n) = 2U_2'(n). \tag{5.4}$$

This formula can also be represented as

$$\widehat{U}_1'(n) = \widehat{U}_2'(n), \tag{5.4$'$}$$

where $\widehat{U}_i'(n)$ denotes the number of unicursal n-edged maps *rooted at an endpoint* and i stands for the number of endpoints. Indeed, reasoning in the same manner as in Sect. 3.1 in the proof of Theorem 4 we see that $\widehat{U}_1'(n) = U_1'(n)/(2n)$ (as a matter of fact, $\widehat{U}_1'(n) = U_1^+(n)$) whereas $\widehat{U}_2'(n) = U_2'(n)/n$.

Is there a direct bijective proof of (5.4)? Note that for *unrooted* maps of the same types the corresponding equality does not hold: $U_1^+(n) \neq 2U_2^+(n)$ according to (3.6) and (3.7).

The same question concerns another curious identity

$$U'(n) = \tfrac{1}{6}(n+1)(n+2)E'(n), \tag{5.5}$$

which follows from formulae (2.1) and (2.13).

The next identity follows from formula (4.3) and the well-known formula of Tutte [23] for the number $S'(n)$ of rooted *non-separable* maps with n edges:

$$B'(2n) = 3^{n-1}S'(n+1). \tag{5.6}$$

It would also be nice to find a direct proof of it.

5.3 Generalizations. It would be interesting to extend the above-mentioned results to obtain formulae for counting rooted maps of the classes we have treated here as a function of other parameters as well as the number of edges, such as the number of vertices, the valency of the vertex containing the root, the sequence of vertex- and/or face-valencies or, in the case of bipartite and bi-eulerian maps, the number of vertices in each part (incidentally, a parametric formula for the number of rooted bipartite maps by number of edges and number of vertices in each part appears in [4, pp.125,126]). It would also be interesting to consider the classes of bipartite unicursal maps necessary for counting unrooted bi-eulerian maps with a given number of edges by the method presented in Section 3.

Numerical results. Tables 1 and 2 contain numerical data for unicursal, eulerian and bi-eulerian maps. The values for $n \leq 6$ can be verified by the Atlas of maps [6] (for some quantities, in fact, we first guessed the formulae from data extracted from the Atlas). Instead of using (4.12) and (4.14) to calculate $E'_{\mathrm{NS}}(n)$ and $B'_{\mathrm{NS}}(n)$, we used Maple to substitute for $f(x)$ from (4.5) and (4.13), respectively, into formula (4.4) and evaluate $g(x)$.

Acknowledgement. We wish to thank Gilles Schaeffer and Cedric Chauve for fruitful discussion of various results concerning the enumeration of eulerian maps and the anonymous referees for careful reading of our manuscript and their useful remarks.

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Table 1: Unicursal maps

n	$U^+(n)$	$U_2^+(n)$	$U'(n)$	$U_2'(n) = U_1'(n)/2$	$\widetilde{U}'(n)$
2	2	1	6	2	5
3	9	3	40	12	28
4	38	11	280	80	168
5	214	62	2016	560	1056
6	1253	342	14784	4032	6864
7	7925	2152	109824	29568	45760
8	51620	13768	823680	219648	311168
9	346307	91800	6223360	1647360	2149888
10	2365886	622616	47297536	12446720	15049216
11	16421359	4301792	361181184	94595072	106502144
12	115384738	30100448	2769055744	722362368	760729600
13	819276830	213019072	21300428800	5538111488	5477253120
14	5868540399	1521473984	164317593600	42600857600	39710085120
15	42357643916	10954616064	1270722723840	328635187200	289650032640
16	307753571520	79420280064	9848101109760	2541445447680	2124100239360
17	2249048959624	579300888960	76467608616960	19696202219520	15651264921600
18	16520782751969	4248201302400	594748067020800	152935217233920	115819360419840
19	121915128678131	31302536066560	4632774416793600	1189496134041600	860372391690240
20	903391034923548	231638727063040	36135640450990080	9265548833587200	6413685101690880

Table 2: Eulerian and bi-eulerian maps

n	$E^+(n)$	$E'(n)$	$E'_{\mathrm{NS}}(n)$	$B'(n)$	$B'_{\mathrm{NS}}(n)$
1	1	1	1		
2	2	3	1	1	1
3	4	12	1		
4	12	56	2	6	2
5	34	288	6		
6	154	1584	19	54	8
7	675	9152	64		
8	3534	54912	230	594	54
9	18985	339456	865		
10	108070	2149888	3364	7371	442
11	632109	13891584	13443		
12	3807254	91287552	54938	99144	4032
13	23411290	608583680	228749		
14	146734695	4107939840	967628	1412802	39706
15	934382820	28030648320	4149024		
16	6034524474	193100021760	18000758	21025818	413358
17	39457153432	1341536993280	78905518		
18	260855420489	9390758952960	349037335	323686935	4487693
19	1741645762265	66182491668480	1556494270		
20	11732357675908	469294031831040	6991433386	5120138790	50348500

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